Math 4670/5670 – Combinatorics
Suggested Exercises for the Semester

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Spring 2022

Last updated: May 9, 2022

These exercises will NOT be collected or graded. They are for your own practice. Just doing the graded homework will definitely not be enough practice to succeed in this course. If you need help with any of these problems, you are welcome to visit office hours!

The problems here will come from our textbook and other sources. I'm sure it's possible to find solutions to most of them online, but I would strongly encourage you not to try. The benefit gained from doing these problems is from the work and struggle involved in solving them, not from just knowing the answer. Plus, other solutions are very likely to use ideas, concept, and notation that we haven't.

All quiz questions will come verbatim from the suggested homework.

At the start of most classes, I will take volunteers to present their solutions to a few problems. We won't have time to do all of them.

Day 2 – Wed, Jan. 26

Covered in class: Chapter 1 of the IBL book up to the top of page 18. Sets, set-builder notation, cardinality, union, intersection.

1. How many even numbers are positive and less than 100? Phrase your answer in terms of the cardinality of a set.

2. Let $S = \{n \in \mathbb{Z} : 1 \leq n \leq 49\}$, i.e., the set of all positive integers from 1 to 49. What is $|S|$? In your own words, why does S have the same number of elements as the set in Problem 1. (Hint: This is asking you to think about a correspondence between the two sets – later we will talk more formally about bijections.)

3. With S as in Problem 2 and

$$T = \{n \in \mathbb{Z} : 0 < n < 100 \text{ and } n \text{ is even}\}$$

(just as in Problem 1), what is $|S \cup T|$.

4. With the same S and T, what is $|S \cap T|$?

Day 3 – Fri, Jan. 28

Covered in class: Chapter 1 of the IBL book, pages 18 and 19. The Sum Principle with examples.

5. Five schools are going to send their baseball teams to a tournament, in which each team must play each other team exactly once. How many games are required?

6. See Figure A.1 on page 198 of the textbook. (The same grid of streets from the Coffee example in class, but with the intersection 1 block south and 1 block west of Coffee closed due to construction.) How many different routes can I take to get coffee if I only go east or north and I avoid that intersection entirely?

Day 4 – Mon, Jan. 31

Covered in class: Chapter 1 of the IBL book, pages 19 and 20. Domino example, Cartesian products.

7. Let $S = \{n \in \mathbb{Z} : -5 \leq n \leq 5\}$ and $T = \{n \in \mathbb{Z} : 1 \leq n \leq 6\}$. Describe the set $S \times T$. What is $|S \times T|$?

8. Let $S_1 = \{1, 2, 3, 4, 5, 6\}$, $S_2 = \{h, t\}$, $S_3 = \{h, t\}$. Describe the set $S_1 \times S_2 \times S_3$. What is $|S_1 \times S_2 \times S_3|$? How is this related to the group work problem from class about rolling a die and flipping coins?

9. A tennis club has $2n$ members. We want to pair up the members to play singles matches. How many ways are there to split $2n$ people into pairs? (Note: the order of the two people in the pair does not matter, and the order you pick the pairs does not matter.)

10. Suppose there are 12 flavors of ice cream at the grocery store, and you want to buy some pints.

    (a) How many ways are there to choose 3 pints of <u>different</u> flavors?

    (b) How many ways are there to choose 2 pints if they don't have to be different flavors?

    (**Edited 2/2/22:** I changed the numbers to make the question a little easier, particularly part (b).)

Day 5 – Wed, Feb. 2

Covered in class: Cartesian products, the product principle.

11. Suppose you flip a coin five times in a row, recording the sequence of heads and tails you see, e.g., (h, h, t, h, t). How many different sequences of flips are possible?

12. Suppose you flip a coin n times in a row and record the outcomes in the same way. Now how many outcomes are possible?

13. A PIN (personal identification number) is a sequence of four digits used for security purposes by banks and order organizations to protect consumer information. Each digit is typically 0 to 9. For example, 0394 is a PIN.

    (a) How many PINs are there?

    (b) How many PINs have no repeated digits?

    (c) How many PINs repeat at least one digit?

Day 6 – Fri, Feb. 4

Covered in class: The product principle, subsets, beginning of functions, pages 22-24.

14. Let A be a finite set with n elements and let B be a finite set with m elements. How many valid functions are there from A to B?

15. Consider all possible subsets of $\{1, 2, 3\}$. How many unordered pairs of distinct subsets (i.e., $\{A, B\}$ with $A \neq B$) are there? Now, among these pairs of distinct subsets, how many are there with the following properties?

    (a) $A \cup B = \{1, 2, 3\}$

    (b) $A \cap B = \{\}$

    (c) $A \cap B = \{3\}$

    (d) $|A \cap B| = 1$

Day 7 – Mon, Feb. 7

Covered in class: Finish Lecture 1 (Functions).

16. Read the section called "organizing data" at te end of Chapter 1 of "Inquiry-Based Enumerative Combinatorics". Think about how it relates to some of the problems we've done in class.

17. Let A be a set of size n. How many functions $f : A \to \{x, y\}$ are there?

18. Let A be a set of size n. How many surjections $f : A \to \{x, y\}$ are there?

19. Let A be a set of size n. How many injections $f : A \to \{x, y\}$ are there?

20. Let A be a set of size n and let B be a set of size m. How many injections $f : A \to B$ are there?

Day 8 – Wed, Feb. 9

Covered in class: Start Lecture 2 (overcounting, quotient principle).

21. Consider the tennis pairs problem in Exercise 9. Suppose that in addition to pairing up the members, you also want to pick which person out of each pair will serve first. How many ways can this whole thing be done (pairing up, then picking who serves)?

22. Suppose that there is a roller coaster with n cars, and each car seats two people. If there are n Marquette students and n UW Milwaukee students in line, in how many ways can those $2n$ people sit in the roller coaster such that each car contains one student from each school? Assume that it is counted as a different seating arrangement if two students in one car switch seats.

23. Whichever way you answered the previous question, try to find a completely different way to approach it that leads you to the same answer (although the answer might be expressed differently).

24. Answer the same question above, but assuming that it is counted as the same seating arrangement if the two students in one car switch seats.

Day 9 – Fri, Feb. 11

Covered in class: Continued Lecture 2 (overcounting, quotient principle).

25. Another tennis problem! Suppose there are $4n$ people, and you are asked to group them up for doubles games. A doubles game consist of four people, two on one team, and two on another team. How many ways can you group $4n$ people into doubles games? (The order of a pair of people on the same team doesn't matter. The order of the two pairs who play against each other doesn't matter. The order of the doubles games doesn't matter.)

26. Answer the same question above if you also have to pick which person on which team serves first for each game.

27. There are 100 students in a gym class. The instructor wants to split them into 5 groups of 20 students each. In how many ways can this be done?

28. What if the instructor also wants to pick one person on each team to be the team captain?

Day 10 – Mon, Feb. 14

Covered in class: Finished Lecture 2 (overcounting, quotient principle).

29. Suppose there are 100 people in a calculus class, and they all got an A on their midterm exam, and every person wants to high-five every other person exactly once. How many high-fives will occur? Note that "Alice high-fiving Bob" also counts as "Bob high-fiving Alice". (Try to answer this question thinking about overcounting.)

30. In how many different ways can the letters of the word `MISSISSIPPI` be rerranged? (Again, think about overcounting!)

Day 11 – Web, Feb. 16

Covered in class: Started Lecture 3 (k-permutations, Chapter 2 in IBL Book).

31. At a concert, you and three friends occupy seats 1, 2, 3, and 4 of row ZZ. How many ways are there for the four of you to sit down in which you take seat 1? How many in which you take seat 3?

32. At a concert, you, Alice, Bob, and 7 additional friends occupy seats 1 through 10 of row ZZ. How many ways are there for the ten of you to sit down in which you take seat 1 and Alice takes seat 4 and Bob takes seat 8?

33. How many 4 digit PINs use the digits $\{2, 7, 9\}$ and no others (so, one digit will be repeated twice)?

Day 12 – Fri, Feb. 18

Exam 1

Day 13 – Mon, Feb. 21

Covered in class: Continued Lecture 3 (k-permutations, Chapter 2 in IBL Book)

34. Suppose ten friends go to a movie, but it is opening weekend and they can't find ten seats in a row. In one row, they find seats 1-6 unoccupied. How many different ways can six of the friends sit together in these seats? (Ignore what happens to the four friends who don't sit in these seats.)

35. Suppose a bank allows its PINs to include letters as well as numbers, but they still have to be length four.

    (a) How many PINs are there, assuming the letters are lowercase? Are these k-permutations? If so, for what k and n?

    (b) How many PINs don't repeat any characters? Are these k-permutations? If so, for what k and n?

    (c) If we allow letters to appear as upper- and lowercase (e.g., 'a' and 'A' are considered different characters), how many PINs are there? Are these k-permutations? If so, for what k and n?

    (d) Allowing both upper and lowercase letters, how many PINs don't repeat any characters? Are these k-permutations? If so, for what k and n?

    (e) Some of the PINs from part (d) have the same letter appearing twice–once as a lowercase letter and once as an upper case letter. For example: "a32A". How many of the PINs from part (d) don't have this property? Are these k-permutations? If so, for what k and n?

36. How many ways are there to arrange a deck of cards so that all the clubs appear before any of the spades, which appear before any of the hearts, which appear before any of the diamonds? Answer this in terms of k-permutations and the notation $P(n, k)$.

Day 14 – Wed, Feb. 23

Covered in class: Continued Lecture 3 (k-permutations, algebraic and combinatorial proofs, Chapter 2 in IBL Book)

37. Give both an algebraic and a combinatorial proof of the identity:

$$P(n, n) = P(n, k) \cdot P(n - k, n - k).$$

    Note: I fixed a typo in the equation at 2pm on Thursday, Feb 24.
    Hint: The left-hand side counts the number of permutations of n things. How does the right-hand side also pick a full order on n things, but in a two-step process?

38. During group work, we answered the problem: how many ways can you play n rooks on an $n \times n$ chessboard so that none of them can attack each other, and we assumed the rooks were identical. How many ways are there to do it if the rooks are considered distinct? (For example, if they are n different colors.) Try to think of as many different ways to answer this question as you can.

39. Same question for placing k rooks on an $n \times k$ board, with $n > k$.

40. Now suppose that you want to place k non-attacking, identical rooks on an $n \times n$ board. First, explain how this can be related to the homework question of assigning distinct pieces of candy to distinct kids, then find a formula for the number of ways to do this.

41. How many ways can we place k non-attacking, distinct rooks on an $n \times n$ board?

42. These questions become a **lot** harder if we replace rooks with queens, who can move any amount vertically, horizontally, or diagonally. There are 92 ways to put 8 non-attacking, identical queens on an 8×8 chessboard. Try to find one!

43. With a lot of work, people have computed the number of ways to put n non-attacking, identical queens on an $n \times n$ board for $n = 1, 2, \ldots, 27$, but nobody knows the answer for 28 or higher. However, a mathematician named Michael Simkin proved in 2021 that even though we don't know the exact number, it is approximately $(0.143n)^n$. If you're interested in learning more, there is a magazine article here: `https://www.quantamagazine.org/mathematician-answers-chess-problem-about-attacking-queens-20210921/`. There is no question here, I just thought you might enjoy seeing how close we are to material that is subject to very active research!

Day 15 – Fri, Feb. 25

Covered in class: Finished Lecture 3 (k-permutations, algebraic and combinatorial proofs, Chapter 2 in IBL Book). Started Lecture 4 (combinations, Chapter 3 in IBL Book)

44. Consider a question from earlier in the semester: You go to the grocery store, and they have 12 flavors of ice cream. How many ways are there to pick 3 different flavors to buy, assuming the order that you pick them does not matter? Our answer at the time was $\frac{12\cdot 11\cdot 10}{3!}$. This number is equal to $\binom{n}{k}$ for what n and what k? (I'm not asking you to know about the formula for $\binom{n}{k}$, I'm asking you to know the definition, and how it relates to this problem.)

45. Suppose a pizza place has 9 possible toppings you can order, and you have a coupon for a large pizza with at most 3 toppings? How many ways are there to choose at most 3 toppings out of the 9? (As you can probably tell, the order you choose them doesn't matter.) Since we don't have a formula yet, give your answer only in terms of $\binom{n}{k}$ for various values of n and k.

Day 16 – Mon, Feb. 28

Covered in class: Continued Lecture 4 (combinations, Chapter 3 in IBL Book)

46. Prove that if n is a positive odd integer, then $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$.

47. Give algebraic and combinatorial proofs of the identity

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

Note: I fixed a typo in the equation at noon on Thursday, March 3. The rightmost term said $\binom{n-k}{m}$ instead of $\binom{n-k}{m-k}$. To see that this original version was wrong, plug in $n = 4$, $m = 3$, $k = 2$. This makes the right-hand side equal to 0.
Hint: For the combinatorial proof, show that both sides count the number of ways to pick a team of m people out of n people total, and then choosing a subgroup of k out of those m people to be the leadership team.

Day 17 – Wed, Mar. 2

Covered in class: Finished Lecture 4 (combinations, Chapter 3 in IBL Book). Barely started Lecture 5 (applications of binomial coefficients).

48. Consider the identity $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. Give an algebraic proof using induction.

49. Consider the identity $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. Give a combinatorial proof. *Hint:* Prove that both sides count the number of ways to start with n people, pick a nonempty group of them of any size, and pick 1 person in the group to be in charge.

Day 18 – Fri, Mar. 4

Covered in class: Continued Lecture 5 (applications of binomial coefficients).

No new suggested homework today. Spend time looking at Homework 3. You should be able to do all Homework 3 questions now, and all of those topics are fair game for Exam 2.

Day 19 – Mon, Mar. 7

Covered in class: Continued Lecture 5 (multinomial coefficients).

50. How many 10 digit PINs can be made out of three 1s, three 2s, two 6s and two 7s? Express your answer as (1) the product of binomial coefficients, and (2) as a single multinomial coefficient.

51. Suppose that 6 people are invited for job interviews? How many different ways are there for 2 of them to be interviewed on Monday, 2 on Wednesday, and 2 on Friday?

52. A company has just hired seven accountants, eight programmers, and two HR managers. The company has four different locations employees can work: North, South, West, and East. They want to put four of the programmers in the North location, the other four can be distributed to the other three locations in any way. Out of the 7 accountants, there must be three at North, one at South, two at West, and one at East. The two HR managers can go anywhere, but they cannot go to the same location as each other. How many different arrangements of these 17 employees are there? (Note that the employees are considered distinct — for example, switching two programmers in different locations is considered a different outcome.)

    (Try to express your answer with one binomial coefficient, one multinomial coefficient, one $P(n, k)$ coefficient, and one exponentiation.)

Day 20 – Wed, Mar. 9

Covered in class: Finished Lecture 5 (stars and bars).

53. Suppose 4 people are running for student body president, and 1000 people vote. How many possible vote totals are there? (This is not considering who actually voted for whom, just the final totals, like 210/492/78/220.)

54. If there are n kids at your door, and you want to distribute exactly m identical pieces of candy to them, in how many ways can you do it such that every kid gets at least one piece?

55. If there are n kids at your door, and you have m Twix bars (the best candy) and k Snickers bars (gross), how many ways can you distribute all of the candy bars to all of the kids?

56. If a grocery store has unlimited amounts of apples, bananas, cantaloupe, and dragon fruit, in how many different ways can you buy 10 pieces of fruit? (Figure out how you can state this as a distribution problem—what are the buckets, what are you putting into the buckets.)

Day 21 – Fri, Mar. 11

Exam 2

Day 22 – Mon, Mar. 21

Covered in class: Started Topic 6 (The Binomial Theorem)

57. What is the coefficient of x^2y^5 in $(2x+3y)^7$?
58. Based on some experiments, what coefficient of $(a+b)^n$ do you think is largest?
59. This exercise will walk you through proving your answer to the previous question.
    (a) Simplify the quantity $\dfrac{\binom{n}{k+1}}{\binom{n}{k}}$.
    (b) Prove that if $0 \le k \le \dfrac{n-1}{2}$, then $\dfrac{\binom{n}{k+1}}{\binom{n}{k}} \ge 1$.
    (c) Conclude that the binomial coefficients $\binom{n}{k}$ get larger and larger (or, at least, don't get smaller) as k increases from 0 until $\left\lfloor \dfrac{n-1}{2} + 1 \right\rfloor$. (The notation $\lfloor x \rfloor$ means "x rounded down to the nearest whole number".
    (d) Use this, together with the fact that $\binom{n}{k} = \binom{n}{n-k}$, to conclude that out of all the binomial coefficients $\binom{n}{k}$ in the nth row of Pascal's triangle, the one in which $k = \left\lfloor \dfrac{n-1}{2} + 1 \right\rfloor$ is at least as large as all the others (it may be tied with another).

Day 23 – Wed, Mar. 23

Covered in class: Continued Topic 6 (The Binomial Theorem)

60. Based on some experiments, what coefficients of $(a+b+c)^n$ do you think are the largest? You do not need to prove your answer.
61. We have previously seen both algebraic and combinatorial proofs of the identity $n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$. Prove it a third way by applying the Binomial Theorem to $(1+t)^n$ and then taking the derivative of both sides with respect to t.

62. Make up your own question similar to the ones from lecture, group work, and suggested homework that requires the Binomial Theorem to solve. Get creative! (Consider swapping problems with a classmate so you can solve each other's problems.)

Day 24 – Fri, Mar. 25

Covered in class: Finished Topic 6 (The Binomial Theorem), Started Topic 7 (Recurrences)

63. Compute $\displaystyle\sum_{k=0}^{100}(-1)^k\binom{100}{k}$.

64. Compute $\displaystyle\sum_{k=0}^{50}(-1)^k\binom{50}{k}(\sqrt{2})^k(\sqrt{3})^{n-k}$.

65. What is the coefficient of a^7 in $\displaystyle\left(\frac{3}{4}a^{2/3}+\frac{2}{3}a^{1/2}\right)^{12}$?

Day 25 – Mon, Mar. 28

Covered in class: Continued Topic 7 (Recurrences)

66. Guess a recurrence for the sequence $1, 3, 7, 15, 31, 63, \ldots$, and check your answer. Can you find an explicit formula?

67. Guess a recurrence for the sequence $1, 2, 3, 6, 11, 20, 37, 68, 125, \ldots$, and check your answer.

68. Let $\lceil x \rceil$ denote the quantity x, rounded up to the nearest integer. Compute the first seven terms of the sequence defined by the recurrence $a_n = n + a_{\lceil n/2 \rceil}$ with initial conditions $a_0 = 3$ and $a_1 = 5$.

69. Compute the first ten terms of the sequence defined by the recurrence $a_n = a_{n-1} - a_{n-2}$ with $a_0 = 1$ and $a_1 = 4$.

Day 26 – Wed, Mar. 30

Covered in class: Continued Topic 7 (Recurrences)

No new suggested homework today. Make some progress on Homework 4 so you can ask questions in office hours before I go out of town!

Day 27 – Fri, Apr. 1

Covered in class: Finished Topic 7 (Recurrences)

70. Let a_n be the number of compositions of n in which all of the parts are just 1 or 2 (for example, the valid compositions of 4 are $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, and $2+2$). Find a recurrence (with initial conditions) for a_n.

71. Let a_n be the number of compositions of n in which all of the parts are just 1, 2, or 3. Find a recurrence (with initial conditions) for a_n.

Day 28 – Mon, Apr. 4

Covered in class: Started Topic 8 (Introduction to Graph Theory)

72. See the graph G in Figure 5.46 on page 102 (in Section 5.9, Exercises) of the free textbook "Applied Combinatorics" by Keller and Trotter.

    (a) What is the degree of vertex 8?

    (b) What is the degree of vertex 10?

    (c) How many vertices of degree 2 are there in G? List them.

73. See Figure 5.48 on page 104. This figure contains four graphs on six vertices. Determine which (if any) pairs of graphs are isomorphic. For pairs that are isomorphic, explain which labels are changed to which. For pairs that aren't isomorphic, explain why.

Day 29 – Wed, Apr. 6

Covered in class: Finished Topic 8 (Introduction to Graph Theory). Started Topic 9 (Trees).

74. See the graph G in Figure 5.46 on page 102 (in Section 5.9, Exercises) of the free textbook "Applied Combinatorics" by Keller and Trotter.

    (a) Find a cycle of length 8 in G.

    (b) What is the length of a shortest path from 3 to 4?

    (c) What is the length of a shortest path from 8 to 7?

    (d) Find a path of length 5 from vertex 4 to vertex 6.

75. Draw a graph with 8 vertices, all of odd degree, that does not contain a path of length 3, or explain why such a graph does not exist.

76. Draw a graph with 6 vertices having degrees 5, 4, 4, 2, 1, and 1 or explain why such a graph does not exist.

77. For the next Olympic Winter Games, the organizers wish to expand the number of teams competing in curling. They wish to have 14 teams enter, divided into two pools of seven teams each. Right now, they're thinking of requiring that in preliminary play, each team will play seven games against distinct opponents. Five of the opponents will come from their own pool and two of the opponents will come from the other pool. Describe the goal in terms of graphs, vertices, and edges, and either draw a graph that achieves the goal or explain why it is not possible.

Day 30 – Fri, Apr. 8

Covered in class: Finished Topic 9 (Trees).

78. Let $T = (V, E)$ be a tree. Prove that adding any additional edge to T (connecting two vertices that already exist in T) must create a cycle.

79. Let $T = (V, E)$ be a tree. Prove that deleting any single edge of T will cause T to become disconnected.

80. Prove that any connected graph with n vertices must have at least $n - 1$ edges. (In other words, it is not possible to build a connected graph with n vertices and $n - 2$ edges or fewer.)

81. We proved in class that every tree with n vertices must have exactly $n - 1$ edges. In the exercise, I want you to prove a version of the converse: Suppose G is a connected graph with n vertices and $n - 1$ edges. Then G must be a tree.

Day 31 – Mon, Apr. 11

Covered in class: Started Topic 10 (Walks in Graphs)

82. Applied Combinatorics book, Chapter 5, Exercise 8

83. Applied Combinatorics book, Chapter 5, Exercise 9 (only first half about Eulerian)

84. Applied Combinatorics book, Chapter 5, Exercise 10

85. Applied Combinatorics book, Chapter 5, Exercise 11

Day 32 – Wed, Apr. 13

Exam 3

Day 33 – Wed, Apr. 20

86. Applied Combinatorics book, Chapter 5, Exercise 9 (second half about Hamiltonian cycles)
87. Applied Combinatorics book, Chapter 5, Exercise 11 (if you did not do it before — prove that a graph has an Eulerian path [the book calls it a trail] if and only if it has at most two vertices of odd degree)
88. Applied Combinatorics book, Chapter 5, Exercise 12

Day 34 – Fri, Apr. 22

89. Applied Combinatorics book, Chapter 5, Exercise 13
90. Applied Combinatorics book, Chapter 5, Exercise 14
91. Applied Combinatorics book, Chapter 5, Exercise 15
92. Find a graph G that *does not arise* as the interval graph corresponding to some meeting requests (i.e., there is no set of meeting requests that lead to the graph G).

Day 35 – Mon, Apr. 25

93. Suppose $G = (V, E)$ is a bipartite graph with parts A and B (i.e., $A \cup B = V$, $A \cap B = \emptyset$, and there are no edges between any two vertices in A or between any two vertices in B). Prove that
$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$
94. A graph is called *regular* if every vertex has the same degree. Prove that if G is a regular graph in which every vertex has degree $d \geq 1$, then every bipartite decomposition of the vertices of G into parts A and B must have the property that $|A| = |B|$.
95. Prove that all trees are bipartite.

Day 36 – Wed, Apr. 27

96. Applied Combinatorics book, Chapter 5, Exercise 30

Day 37 – Fri, Apr. 29

No suggested homework today.

Day 38 – Mon, May 2

97. Applied Combinatorics book, Chapter 5, Exercise 28
98. Applied Combinatorics book, Chapter 5, Exercise 29
99. Applied Combinatorics book, Chapter 5, Exercise 32

Day 39 – Wed, May 4

100. Devise a map that obeys our caveats but requires four colors to color.

Day 40 – Fri, May 6

101. Applied Combinatorics book, Chapter 10, Exercise 1
102. Suppose you draw 5 cards from a deck of 52, without replacement. Define the sample space S to be all <u>unordered</u> five-card hands. What is $|S|$? Consider the event of drawing five cards that all have different values. Write this as an event E and find $\mathbb{P}(E)$.
103. Suppose a jar contains 5 red marbles, 3 blue marbles, and 8 green marbles, and that you randomly pick one marble in your left hand and one marble in your right hand (without replacement). Define S and state $|S|$. Consider the event that both marbles in your hands are green. Write this as an even E and find $\mathbb{E}$.
104. Consider the example from class in which we roll two fair six-sided dice. In that example, we preferred to define S to be all ordered pairs in $\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$. That gives $|S| = 36$ and every outcome is equally likely. In this question, I want you to consider the version where you define S to be all <u>unordered</u> pairs, so

$$S = \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 2\}, \ldots\}.$$

Prove that $|S| = 21$. (How could you find this with stars and bars?) Show that the outcomes are S are NOT equally likely. For example, the probability of the outcome $\{1, 1\}$ is smaller than the probability of the outcome $\{1, 2\}$.

Day 41 – Mon, May 9

105. Suppose you have a jar of marbles with 6 red marbles, 9 blue marbles, and 5 green marbles. If you draw three marbles without replacement, what is the probability that two of them are red and one of them is blue?