MSCS 6040 – TAKE HOME EXAM

assigned: Wednesday, March 6
due: Wednesday, March 20

Instructions:

1. Please write your work neatly and clearly.
2. You must explain all reasoning. It is not sufficient to just write the correct answer. Correct answers with no justification will receive no credit.
3. You must submit your completed exam on D2L by **11:59pm on Wednesday, March 20**. You may submit any combination of images, pdfs, and mlx files, although it should be clear to me which document corresponds to which exercise, as *D2L tends to randomize the order of submitted documents*.
4. For any questions that ask you to write code and produce output, you **must** submit an mlx file so I can run your code and verify the output.
5. **RULES FOR OUTSIDE SOURCES:**
   It is permissible to use:
   (a) Either of our textbooks, or any other published books or articles.
   (b) *Your* course notes.
   (c) The internet for base-level questions: Matlab synatax, basic definitions of terms, etc.
   (d) Me, via email or office hours.

   It is not permissible to use:
   (a) Any classmates, other students, other professors, etc.
   (b) Anybody else's course notes (it is okay if you had previously borrowed someone else's notes because you missed a day, etc).
   (c) The internet for anything remotely approaching the actual question asked.

   For example, it is okay to search the internet for "How to compute the SVD in Matlab", but it is not okay to search the internet for "Examples of numerical instability in Matlab".

   If you are in doubt about what is okay, email me!

   It is abundantly easy to figure out when an answer has been copied from another source. Cheating on this exam will result in a score of 0 for the whole exam.

The Marquette University honor code obliges students:

- To fully observe the rules governing exams and assignments regarding resource material, electronic aids, copying, collaborating with others, or engaging in any other behavior that subverts the purpose of the exam or assignment and the directions of the instructor.
- To turn in work done specifically for the paper or assignment, and not to borrow work either from other students, or from assignments for other courses.
- To complete individual assignments individually, and neither to accept nor give unauthorized help.
- To report any observed breaches of this honor code and academic honesty.

1. Give an example of numerical instability in Matlab, different from the ones we've seen in class. You do not need to say anything technical about condition numbers. Demonstrate the instability by showing that small input changes can lead to relatively large output changes.

2. Define the 10×5 matrix A and the 10×1 vector b by

$$A = \begin{bmatrix} -2 & 1 & 3 & 3 & 6 \\ 3 & 6 & 2 & -5 & 6 \\ 0 & -3 & 7 & 9 & 7 \\ 9 & 3 & -8 & -2 & 4 \\ 7 & 9 & 6 & -6 & -3 \\ -9 & 8 & 8 & 9 & 2 \\ 5 & -4 & -3 & -3 & -8 \\ -6 & 6 & 4 & -2 & 7 \\ -8 & -4 & 0 & 7 & 4 \\ 3 & 1 & 2 & -2 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -7 \\ 5 \\ 9 \\ 6 \\ -2 \\ 3 \\ 5 \\ -8 \\ -8 \end{bmatrix}$$

   (a) Solve $Ax = b$ using least squares via each of the three methods outlined in Lecture 11. Use the matlab command "format long" to display your results to 16 decimal places. When using the QR factorization method, use your implementation of Algorithm 8.1 from homework.

   (b) Compare your results.

   (c) What is the orthogonal projector matrix P that maps b to Ax?

   (d) Verify by calculation that $P^2 = P$ and $P^* = P$, and then explain in words why this should be true given the fact that P maps any vector $\widehat{b}$ to the vector $\widehat{y}$ such that $\widehat{y}$ is the point in Range(A) closest to $\widehat{b}$.

3. (a) Devise and implement an algorithm to numerically approximate the induced p-norm of a matrix A. Explain how your algorithm works, including any pictures/plots that you think are helpful.

   (b) Use your algorithm to approximate the p-norm of each of the matrices below for $p = 1, 2, 4, 10, 100$.

      i. $A = \begin{bmatrix} 7 & 2 & 7 \\ 3 & 1 & -6 \\ 6 & 3 & -7 \end{bmatrix}$

      ii. $B = \begin{bmatrix} 6 & -8 & 0 & 9 \\ 3 & 8 & 1 & 5 \\ -7 & 0 & 2 & 1 \\ -7 & -9 & 6 & -8 \end{bmatrix}$

      iii. $C = \begin{bmatrix} 3 & 8 & -7 & -6 & -9 \\ -6 & -5 & -6 & -5 & -7 \\ 8 & 7 & 8 & 7 & 8 \\ 9 & -8 & 4 & 7 & -4 \\ -8 & 3 & -2 & -8 & -5 \end{bmatrix}$

   (c) Matlab only lets you compute matrix p-norms for $p = 1, 2, \infty$, so compare your answers to the 1- and 2-norm computed by Matlab.

4. (Trefethen-Bau, Exercise 12.3) The goal of this problem is to explore some properties of random matrices. Your job is to be a laboratory scientist, performing experiments that lead to conjectures and more refined experiments. Do not try to prove anything. Do produce well-designed plots, which are worth a thousand numbers.

   Define a random matrix to be an $m \times m$ matrix whose entries are independent samples from the real normal distribution with mean zero and standard deviation $m^{-1/2}$. (In Matlab: `A = randn(m,m)/sqrt(m)`.) The factor $\sqrt{m}$ is introduced to make the limiting behavior clean as $m \to \infty$.

(a) What do the eigenvalues of a random matrix look like? What happens, say, if you take 100 random matrices and superimpose all of their eigenvalues in a single plot? If you do this for $m = 8, 16, 32, 64, \ldots$, what pattern is suggested? How does the spectral radius $\rho(A)$ (Exercise 3.2) behave as $m \to \infty$?

(b) What about norms? How does the 2-norm of a random matrix behave as $m \to \infty$? Of course, we must have $\rho(A) \leq \|A\|$ (Exercise 3.2). Does this inequality appear to approach as equality as $m \to \infty$?

(c) What about condition numbers (see the section for the definition of the condition number of a matrix)—or more simply, the smallest singular value $\sigma_{\min}$? Even for fixed m this question is interesting. What proportions of random matrices in $\mathbb{R}^{m \times m}$ seem to have $\sigma_{\min} \leq 2^{-1}, 4^{-1}, 8^{-1}, \ldots$? In other words, what does the tail of the probability distribution of smallest singular values look like? How does the scale of all this change with m?

(d) How do the answers to (a)–(c) change if we consider random triangular instead of full matrices, i.e., upper-triangular matrices whose entries are samples from the same distribution as above?

5. *For this question (and only this question) you may use the internet as extensively as you would like, but* ***do not just copy from the internet!***

   Find an interesting application of numerical linear algebra using techniques we've learned that is not one of the demos from class (image compression, background removal of videos). Describe the application, describe how it incorporates topics from class, and give an example of the application.