MATH 60 – HOMEWORK 8

due Wednesday, May 23

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit. Every step of your answers must be fully justified to receive credit.

It is considered cheating and a violation of the Honor Code to look for answers to these problems on the internet.

You may use a z-table on any question.

Feel free to use a calculator for any basic arithmetic.

1. Suppose a stranger approaches you with two envelopes containing money, with one envelope containing n times as much as the other, where n is any positive integer greater than one. He gives you one at random and allows you to look inside the envelope. He then asks if you would like to switch. Suppose also that you know the probability distribution $p_a = P(\text{smaller envelope contains } a \text{ dollars})$. Find an inequality in terms of p_a and $p_{a/n}$ that determines when you expect to benefit from switching.

2. A machine that bags Skittles is supposed to distribute exactly 100 skittles per bag. It occasionally makes slight errors – it dispenses exactly 100 Skittles 70% of the time, 99 Skittles 15% of the time, 98 Skittles 10% of the time, and 101 Skittles 5% of the time.

   (a) Estimate the probability that if you buy 100 bags of Skittles, you will have less than or equal to 9950 Skittles.

   (b) Estimate the probability that if you buy 100 bags of Skittles, you will have exactly 9950 Skittles.

   (c) Estimate the probability that if the Skittles factory packages 100 bags of Skittles, they will have dispensed strictly more than $10,000$ Skittles.

3. One gram of a certain radioactive material emits electrons at the average rate of 4 per minute according to an exponential distribution. You have sixty separate one-gram samples and you need to test them all to make sure they're still radioactive. So, you hold a Geiger counter up to the first one until an electron is emitted, then move it to the second sample until that one emits an electron, and so on. Assume that you move the Geiger counter from one sample to the next instantaneously and that your reaction time is infinitely good.

   Use the Central Limit Theorem to estimate the probability that this procedure takes you more than 20 minutes.

4. (a) Let $X = \text{Unif}(1,3)$. Find the moment generating function $g_X(t)$.

   (b) Define $Y = \text{Unif}(0,1)$ and note that $g_Y(t)$ is computed in the textbook. Use $g_Y(t)$ to find a simple formula for the nth moment of Y. (*Hint:* It may help to expand $g_Y(t)$ into a formal power series. Don't use L'Hôpital's rule.)

   (c) Note that $X = 2Y + 1$. Use your answer to (b) to compute the nth moment μ_n of X. Your answer will probably have a summation. You may use the *Binomial Theorem*, which says that for a positive integer n

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

5. Let Z be the discrete random variable defined by

$$\mathbb{P}(Z = k) = \begin{cases} \dfrac{d-1}{d^{k-1}}, & k \in \mathbb{N} \text{ and } k \geq 2 \\ 0, & \text{otherwise} \end{cases}.$$

   (You may assume that this is in fact a proper probability distribution function.) Find the moment generating function of Z and use it to find $\mathbb{E}[Z]$.