MATH 60 – HOMEWORK 3

due Friday, April 20

Instructions: This assignment is due at the beginning of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit. Every step of your answers must be fully justified to receive credit.

It is considered cheating and a violation of the Honor Code to look for answers to these problems on the internet.

1. Suppose that you roll d dice, each with n sides. Let X_1 be the random variable for the smallest value that comes up. For an arbitrary $s \in \{1, 2, \ldots, n\}$, find $\mathbb{P}(X_1 = s)$.

2. Suppose each of n balls labeled 1 to n is placed in one of n boxes labeled 1 to n. Assume the n placements are made independently and uniformly at random (so each box can contain more than one ball). A match occurs at place k if ball number k falls in box k. Find:

   (a) the probability of a match in box i and no match in box j, assuming $i \neq j$;

   (b) the expected number of matches.

3. A building has 15 floors above the basement. If 18 people get into the elevator at the basement, and each chooses a floor at random to get out, independently of the others, at how many floors do you expect the elevator to make a stop to let out one or more of these 18 people?

4. A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second).

   (a) Show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

   (b) Are X and Y independent?

   (c) Why are your answers to (a) and (b) interesting?

5. Imagine the game show "Let's Play a Deal", but this time Monty Hall has had too much to drink. The game goes like this: As usual the contestant picks a door. In the normal game, Monty purposefully opens a different door to reveal a goat. In this game, however, Drunk-Monty just randomly opens one of the other doors without knowing what's behind it. If he accidentally reveals the car, they stop shooting, reset the game, and try again. If he reveals a goat, the game proceeds as normal: the contestant can choose to switch or stay. In this game, does switching still give you an advantage? Or does it give you a disadvantage? Or are the odds now even?

6. You have n ping pong balls labeled $1, 2, \ldots, n$ in a hat. You draw one at random, then you paint it and all other ping pong balls with value less than it red, and you paint all other ping pong balls with value greater than it green. Then you put them all back in a hat. Next, you draw one at random again. What is the expected number of balls painted the same color as the one you drew (including the one you drew)? You may use the following formulas to simplify your answer:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

7. You have 80 dollars and play the following game. A box contains two white balls and two black balls. You draw the balls out one at a time without replacement until all the balls are gone. On each draw, you bet half of your present fortune that you will draw a white ball. If you're correct, you win the amount that you bet, and if you're wrong you lose your whole bet. What is your expected final fortune?