Ten Tips for Writing Mathematical Proofs

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1. Determine exactly what information you are given (also called the *hypothesis*) and what you are trying to prove (the *conclusion*). The hypothesis often appear in the statement after the words **let**, **suppose**, **assume** and **if**. This is information that you can assume is true and use freely. To find the conclusion, look for the words **show** or **then**. Most statements that we will prove in this course follow the following format:

   Let (insert hypothesis). **If** (insert hypothesis), **then** (insert conclusion).

   A statement containing the phrase 'if and only if' is a little different. Think of this as a two statements put into one. To prove a statement of the form 'A if and only if B', you must prove two things:

   1. **If** A **then** B. In other words, use A as the hypothesis and B as the conclusion.
   2. **If** B **then** A. In other words, use B as the hypothesis and A as the conclusion.

2. When writing a mathematical proof, you must start with the hypothesis and via other mathematical truths – such as definitions, theorems or computations – arrive at the desired conclusion. If you get stuck, it is often helpful to turn to definitions. In mathematics we use definitions as tools. When you apply a definition, you generally will get something useful. For example, consider the definition of divisibility:

   Suppose that a and d are integers. Then $d|a$ if and only if there is an integer k such that $a = dk$.

   Whenever you apply the definition of divisibility, you *get* an integer k that you did not have before! So using the definition of divisibility gives you a new object (in this case, an integer) to use. Notice that the definition is an 'if and only if' statement. That means that if you can find an integer k so that $d = ak$, then $a|d$.

3. You want to write proofs in a natural, step-by-step order, like a manual. The proof should begin with stating the assumptions you are using and work logically from that point. It is very frustrating for your reader if steps in the proof are skipped, so even if it seems simple or obvious you should include it. Like a good lawyer, you want to convince your reader that the proof is true beyond a reasonable doubt.

4. Do not use symbols, such as $\Rightarrow$ when you write a proof. This is ambiguous and your reader will not be sure what it means. If you find yourself struggling to get rid of arrows, read your proof out loud. Instead of $\Rightarrow$ you will probably say things like 'then' or 'therefore'. Use these words rather than arrows.

5. Every proof is a small composition. It should have correct grammar, punctuation, spelling and sentence structure. If you have mathematical computations in your proof, write only one equation at a time. A proof should not consist of a long line of computations. Break up computations with phrases such as:

   Since $x + 2 = 4$ it follows that $x = 2$.
   Via algebraic computations, $x + 2 = 4$ and therefore $x = 2$.

   Also, it is considered bad form to begin a sentence with a mathematical symbol or equation. So rather than beginning a sentence with

   $x + 2 = 4$ implies. . . ,

   write instead

   The equation $x + 2 = 4$ implies. . . .

6. Most mathematical proofs do not use the pronoun 'I'. Most proofs, when necessary, use 'we', referring to the writer and the reader. So it is OK to say things like:

   We see that . . .
   We have shown that . . .
   We get . . . .

7. If you introduce a new letter or notation make sure you tell the reader what it is. There should be no undefined variables. Think of your proof as a novel and the variables as characters in the novel. If someone new shows up, introduce them and describe their qualities (e.g. an integer, a natural number, not zero, etc.). Another common error is to introduce *too many* variables. Keep your list of characters as small as possible to help your reader remember.

8. Make the end of your proof obvious to the reader. You can do this by stating things like

   . . . and thus it is proved.
   Therefore we have proved that. . . .

   Make sure that you reach the correct conclusion. It is common to see a small box, $\square$, at the end of a proof or the letters *Q.E.D.* (quod erat demonstrandum, which is Latin for "which was to be shown").

9. Above all, practice and be patient. Proofs are difficult and it takes time to become comfortable writing them. Each person will have their own proof-writing style. Aim to be clear and concise and do not worry if your proof looks very different from others that you see. Always re-read over your proofs to check that the proof makes sense mathematically and grammatically.

10. Here is an example of a proof containing several common errors and another proof that presents the same mathematical steps, but is written in a much better style.

    Statement: Let a, b be integers. If $a|b$ and $2|a$, then $2|b$.

    Not-so-good proof: Let a, b be integers.

    $$\left.\begin{array}{l} a|b \Rightarrow b = ak \\ 2|a \Rightarrow a = 2j \end{array}\right\} \Rightarrow b = 2jk.$$

    So $2|b$.

    Better proof: Let a, b be integers. By hypothesis, $a|b$ and $2|a$. Then the definition of divisibility tells us that there is an integer k so that $b = ak$ and another integer j such that $a = 2j$. By substituting $a = 2j$ into the equation $b = ak$, we get $b = 2jk$. Since j, k are both integers, the product jk is also an integer. Therefore by the definition of divisibility again, we have shown that $2|b$. □

If you are interested in other proof-writing help, I suggest *How to Read and Do Proofs*, by Daniel Solow (many editions exist).