NAME : Key

Math 60

Quiz 2
May 8, 2018

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem.

$\boxed{\textit{You must show all work to receive credit.}}$

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	10	
2	5	
3	5	
4	8	
5	12	
Total	40	

Section 1: True/False/Neither.

1. (10) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

Hint: It's never neither.

(a) If $X = \text{Pois}(\lambda)$, then $\mathbb{E}[X^2] = \lambda^2 + \lambda$.

$\left.\begin{array}{l}\mathbb{E}[X] = \lambda \\ Var(X) = \lambda\end{array}\right\}$ from class

$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

True   False   Neither

(b) If Y is a continuous random variable with cumulative density function $F_Y(y)$, then

$$\lim_{y \to -\infty} F_Y(y) = 0 \qquad \text{and} \qquad \lim_{y \to \infty} F_Y(y) = 1.$$

True   False   Neither

(c) The function

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \sqrt{x/3}, & 0 \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}.$$

continuous ✓
goes to 0 ← ✓
goes to 1 → ✓

is the cumulative density function for some random variable.

True   False   Neither

(d) A negative binomial distribution is a sum of ~~exponential~~ geometric distributions.

True   False   Neither

(e) The sum of two independent binomial distributions is itself a binomial distribution.

only if same p

True   False   Neither

Section 2: Shorter Answer. *You must justify all work.*

2. (5) Suppose that N balls are thrown into 1000 boxes. Each throw is independent, and each ball is equally likely to land in any box. Suppose further that this experiment is repeated a very large number of times, and that you observe that the mean number of empty boxes at the end is 20. Use the concept of Poisson approximation to find N. (You do not need numerical answer.)

Binomial dist for # of balls in a given box = $\text{Bin}(N, \frac{1}{1000})$.

Poisson approx: $\lambda = Np = \frac{N}{1000}$.

Let X = # of balls in a fixed box.

Observation that mean # of empty boxes is 20 $\Rightarrow P(X=0) = \frac{20}{1000} = \frac{1}{50}$.

Let $Y = \text{Pois}(N/1000)$. So,

$$P(X=0) = \frac{1}{50} = P(Y=0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-N/1000}$$

$$\Rightarrow \frac{1}{50} = e^{-N/1000}$$

$$\Rightarrow \ln\left(\frac{1}{50}\right) = -N/1000$$

$$\Rightarrow \boxed{N = 1000 \ln(50)} \approx 3912.$$

3. (5) Bridies' Bearing Works manufactures bearing shafts whose diameters are normally distributed with parameters $\mu = 1$ and $\sigma = 0.002$. The buyer's specifications require these diameters to be within 0.003 centimeters of 1 centimeter.

(a) What fraction of the manufacturer's shafts are likely to be rejected?

(b) If the manufacturer improves her quality control, she can reduce the value of σ. What value of σ will ensure that no more than 1 percent of her shafts are likely to be rejected?

You may use the z-table attached to the back of this quiz.

(a) z-score of $1+0.003$ is $\frac{1.003-1}{.002} = 1.5$.

z-score of $1-0.003$ is $\frac{0.997-1}{.002} = -1.5$.

$$\mathbb{P}(\text{rejection}) = \underbrace{\mathbb{P}(N(0,1) \le -1.5) + \mathbb{P}(N(0,1) \ge 1.5)}_{\text{equal, by symmetry}}.$$

≈ 0.0668, by z-table.

Hence, $\mathbb{P}(\text{rejection}) \approx 2 \cdot 0.0668 = \boxed{0.1336}$

(b) We want to set σ such that

$$\mathbb{P}\left(N(0,1) \le \frac{-0.003}{\sigma}\right) \approx 0.5\% = 0.005.$$

By the z-table, $\mathbb{P}(N(0,1) \le -2.57) \approx 0.005$.
(or -2.58)

Hence $-2.57 = \frac{-0.003}{\sigma} \Rightarrow \boxed{\sigma = \frac{0.003}{2.57}.}$ or 2.58.

We want $\sigma \le \frac{0.003}{2.57}$.

Section 3: Longer Answer.

If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

You must fully justify all steps!

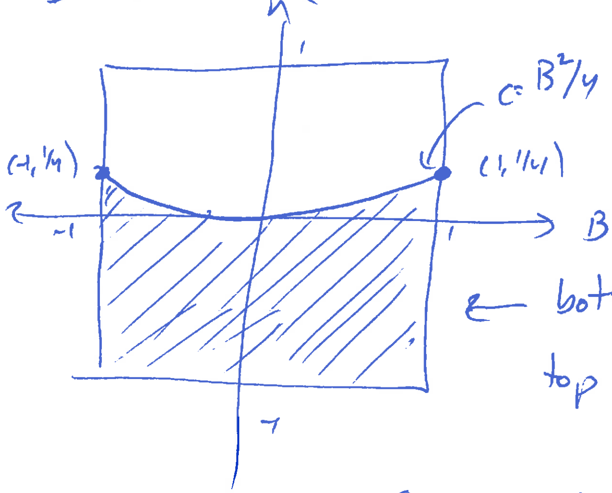
4. (8) Let B and C be independent uniform random variables on the interval $[-1, 1]$. Consider the equation $x^2 + Bx + C = 0$. Find the probability that this equation has two real solutions. *Hint:* The equation has two real solutions if and only if $B^2 - 4C \geq 0$.

$B^2 - 4C \geq 0$

iff $B^2 \geq 4C$

iff $C \leq \frac{B^2}{4}$

Draw a picture!



shaded area $=$ $B^2 - 4C \geq 0$

total area $= 4$.

bottom has area 2

top has area $\int_{-1}^{1} \frac{b^2}{4}\, db = \frac{b^3}{12}\Big]_{-1}^{1} = \frac{1}{6}$.

So, shaded area $= 2 + \frac{1}{6} = \frac{13}{6}$.

$$\mathbb{P}(B^2 - 4C \geq 0) = \frac{\text{shaded area}}{\text{total area}} = \frac{13/6}{4} = \boxed{\frac{13}{24}}$$

5. (12) Emergency loudspeakers have exponentially distributed lifetimes with mean 24 months. To be on the safe side, a town replaces a loudspeaker if it fails before 24 months, or replaces it at 24 months even if it hasn't failed yet. Let L be the random variable for the lifetime of a loudspeaker that is never replaced, and let U be the random variable for how long a loudspeaker is in service under this town's strategy.

$L = \mathrm{Exp}(1/24)$

just like HW 6, #4! If you did that problem before the quiz, you had a big head start.

(a) State the pdf $f_L(x)$ and the cdf $F_L(x)$.

From class:

$$\boxed{f_L(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{24} e^{-x/24}, & x \geq 0 \end{cases}} \qquad \boxed{F_L(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/24}, & x \geq 0 \end{cases}}$$

(b) Find the piecewise formula for the cdf $F_U(x)$ and then sketch it.

$U = \min(L, 24)$. So, if $x \leq 24$ then

$$\mathbb{P}(U \leq x) = \mathbb{P}(L \leq x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-x/24}, & 0 \leq x \leq 24 \end{cases}$$

If $x \geq 24$, then $\mathbb{P}(U \leq x) = 1$.

Hence: $F_U(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-x/24}, & 0 \leq x \leq 24 \\ 1, & x > 24 \end{cases}$

sketch:

(c) Why is the pdf $f_U(x)$ weird, and unlike any pdfs we've seen before?

The cdf has a jump discontinuity, so we can't write down a pdf with our normal methods. (In reality, you need something called a Dirac delta.)

(d) Find $\mathbb{E}[L]$ and $\mathbb{E}[U]$.

Hint: $\int xe^x\,dx = e^x(x-1)+C$

From class: $\mathbb{E}[L] = \frac{1}{1/24} = 24.$

(also from problem statement)

If you spent more than 2 seconds on this part, you didn't use your time wisely!

We can't use the normal formula for U because the pdf doesn't make sense. Instead, define

$$U_1 = \begin{cases} L, & x \le 24 \\ 0, & \text{otherwise} \end{cases} \qquad U_2 = \begin{cases} 0, & x \le 24 \\ 24, & \text{otherwise} \end{cases}$$

So $U = U_1 + U_2$, and thus $\mathbb{E}[U] = \mathbb{E}[U_1] + \mathbb{E}[U_2]$.

Now the normal formula works for U_1 and U_2.

$$\mathbb{E}[U_1] = \int_0^{24} x f_L(x)\,dx = \int_0^{24} \frac{x}{24} e^{-x/24}\,dx = -\int_0^{24} \frac{-x}{24} e^{-x/24}\,dx$$

(u-sub or chain rule)

$$= \left[24 e^{-x/24}\left(-x/24 - 1\right)\right]_0^{24} = \frac{24}{e}(-2) - (24)(-1) = 24 - \frac{48}{e}$$

$$\mathbb{E}[U_2] = \int_{24}^{\infty} 24 \cdot f_L(x)\,dx = \int_{24}^{\infty} 24 \cdot \left(\tfrac{1}{24} e^{-x/24}\right)dx = \left[-24 e^{-x/24}\right]_{24}^{\infty}$$

$$= \frac{24}{e}.$$

So, $\mathbb{E}[U] = 24 - \frac{48}{e} + \frac{24}{e} = \boxed{24 - \frac{24}{e}} \approx 15.17.$

(e) Why is the assumption that the lifetime is exponentially distributed incompatible with the town's replacement scheme?

The exponential distribution is memoryless, so if the loudspeakers lifetime is indeed exponential, then after 24 months its expected additional lifetime is still another 24 months!

Standard Normal Probabilities

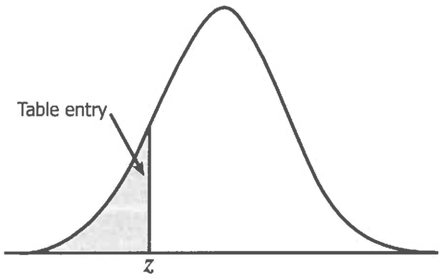


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

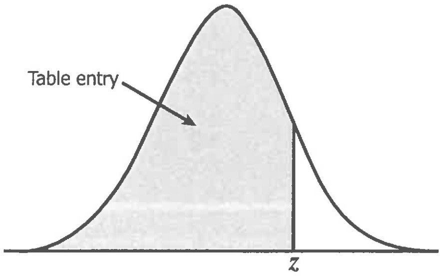


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998