NAME : Key

Math 60

Quiz 1
April 10, 2018

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem.

You must show all work to receive credit.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	10	
2	5	
3	5	
4	10	
5	10	
Total	40	

Section 1: True/False.

1. (10) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) For any events A and B,

$$\mathbb{P}(A \cup B) + \mathbb{P}(\overline{A}) \geq 1.$$

True   False

(b) If S and T are mutually exclusive (disjoint), then they are independent.

True   False

(c) The number of ways to paint a row of n (distinguishable) houses either red or blue is 2^n.

True   False

(d) Suppose that $B_1, B_2, \ldots, B_k$ is a partition of a finite sample space Ω and let $A \subseteq \Omega$. Then,

$$\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \cdots + \mathbb{P}(A|B_k)\mathbb{P}(B_k).$$

True   False

(e) Let A be an event in a finite sample space Ω. Then, A and $\overline{A}$ are independent.

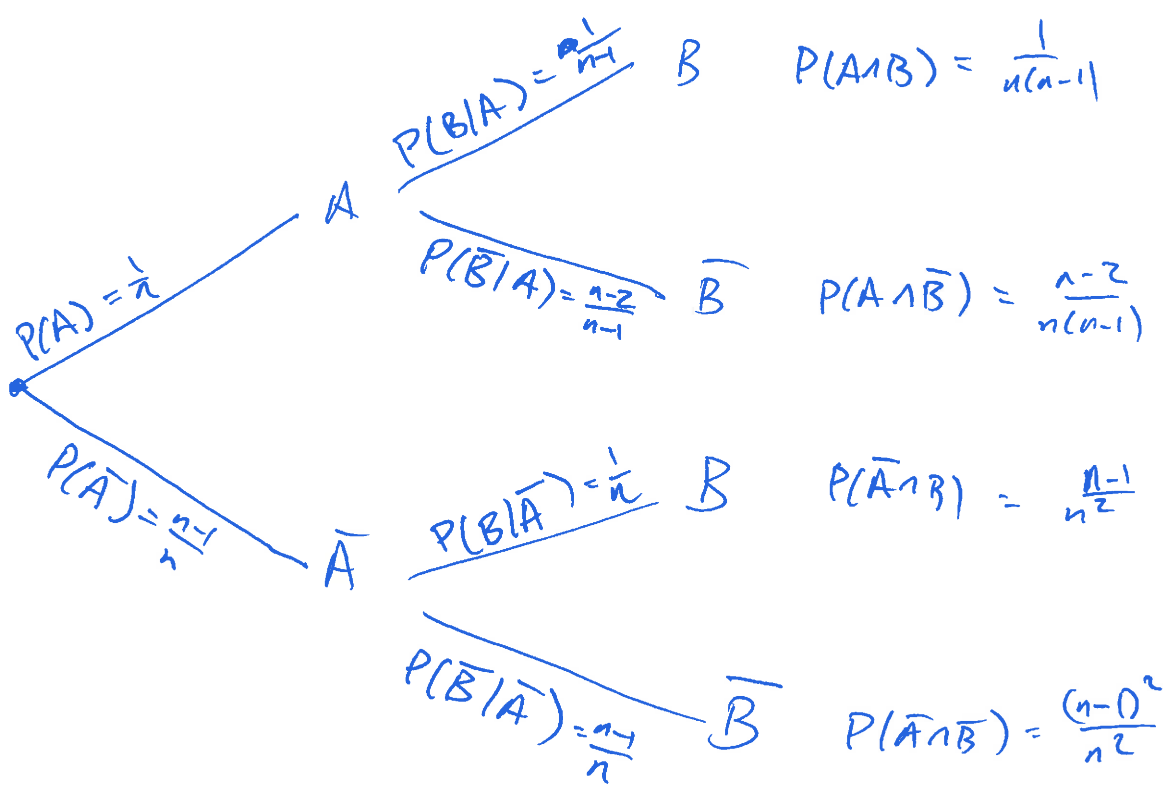
True   False

Section 2: Shorter Answer. *You must justify all work.*

2. (5) You have a bucket of n marbles that are labeled with the integers from 1 to n, but are otherwise identical. Assume $n \geq 10$. You pick one out at random (all outcomes equally likely). If it is marble 5, you keep it out; otherwise you put it back. Then you draw a second one. What is the probability that it is marble 8?

Let $A =$ "first marble is 5".

Let $B =$ "second marble is 8".



$$\mathbb{P}(B) = \frac{1}{n(n-1)} + \frac{n-1}{n^2} = \frac{n}{n^2(n-1)} + \frac{(n-1)^2}{n^2(n-1)} = \frac{n + (n^2 - 2n + 1)}{n^2(n-1)}$$

$$= \frac{n^2 - n + 1}{n^2(n-1)}$$

3. (5) A group of n friends just got into Dartmouth. If each person is assigned randomly to one of Dartmouth's 20 dorms, and assuming that all dorm assignments are equally likely, what is the probability that there are at least two friends in the same dorm?

$$\mathbb{P}(\geq 2 \text{ in one dorm}) = 1 - \mathbb{P}(\text{all in separate dorms})$$

$$\mathbb{P}(\text{all in separate dorms}) = \frac{20 \cdot 19 \cdot 18 \cdot \ldots \cdot (20-n+1)}{20^n}$$

$$= \frac{20^{\underline{n}}}{20^n}$$

So, $\mathbb{P}(\geq 2 \text{ in one dorm}) = 1 - \frac{20^{\underline{n}}}{20^n}$.

Notice that this formula even gives the correct answer, 1, when $n > 20$.

Section 3: Longer Answer.

If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

You must fully justify all steps!

4. (10) Suppose you have N urns lined up, each one contain r red balls and g green balls. Let d be a positive integer. Assume any ball in an urn has an equally likely chance to be drawn. You perform the following procedure:

- ◇ You draw a ball at random from Urn 1. You drop it in Urn 2, along with $d-1$ more of the same color.
- ◇ You then draw a ball at random from Urn 2. You drop it in Urn 3, along with $d-1$ more of the same color.
- ◇ ... (Repeat with each urn down the line.)
- ◇ You then draw a ball at random from Urn $(N-1)$. You drop it in Urn N, along with $d-1$ more of the same color.

Now, draw a ball at random from Urn N. Prove by induction that the probability this ball is green is $\frac{g}{r+g}$. (You may use the next page for more space.)

Proof By Induction

Base case, $N=1$: Drawing from the first urn has a probability of $\frac{g}{r+g}$ for green because the are g green balls, $r+g$ total, and each ball is equally likely to be drawn. ✓

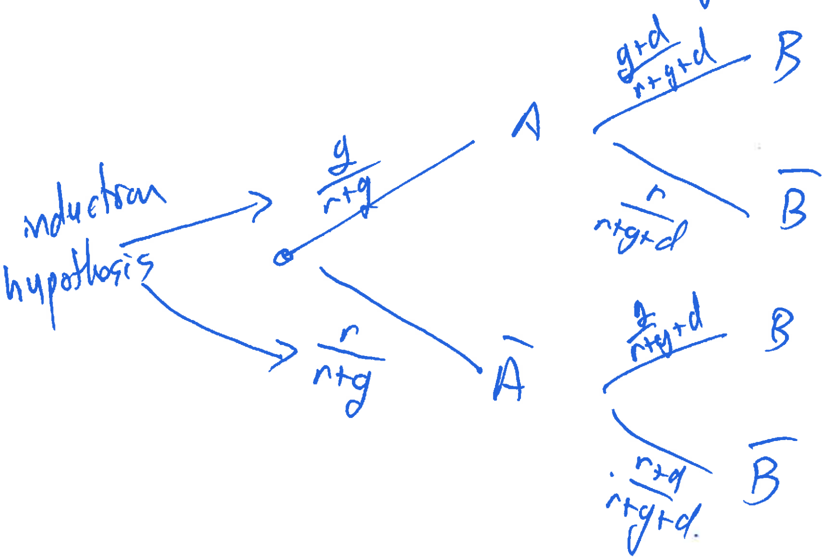
Inductive Step: Assume $P(k)$ = "after repeating this process with k urns, the prob. of drawing green from urn k is $\frac{g}{r+g}$".

(more space for Q3)

We will now prove $P(k+1)$. Suppose we carry out the process for the first k urns.

Let $A =$ "draw g out of urn k"

Let $B =$ "draw g out of urn $k+1$".



So, $$\mathbb{P}(B) = \frac{g}{r+g} \cdot \frac{g+d}{r+g+d} + \frac{r}{r+g} \cdot \frac{g}{r+g+d}$$

$$= \frac{g^2+gd+gr}{(r+g)(r+g+d)} = \frac{g\cancel{(r+g+d)}}{(r+g)\cancel{(r+g+d)}}$$

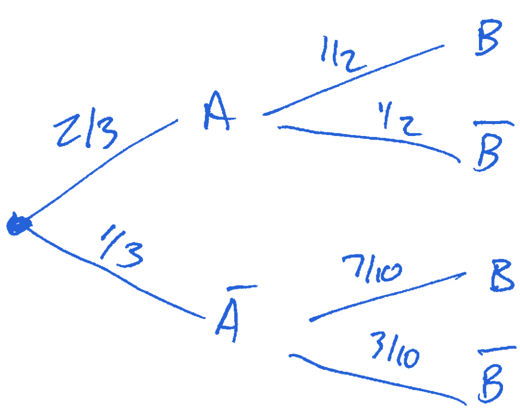
$$= \frac{g}{r+g}. \quad \blacksquare$$

5. (10) A company produces two different types of assortments of candy. Their two biggest sellers are "Equal Split" which is 50% Twix and 50% Kit-Kat (by quantity) and "Uneven Break" which is 30% Twix and 70% Kit-Kat. A store buys two boxes of Equal Split and one box of Uneven Break. Unfortunately, in transit the three boxes lost their labels. In an effort to determine which box is which, the delivery man chooses one of the three boxes at random and pulls from that box one piece of candy (at random).

(a) What is the probability that the piece of candy chosen was a Kit-Kat?

Let A = "chose ~~an~~ equal split"

Let B = "chose ~~_____~~ kitkat"



$$\mathbb{P}(B) = \frac{2}{3}\cdot\frac{1}{2} + \frac{7}{10}\cdot\frac{1}{3}$$

$$= \frac{1}{3} + \frac{7}{30} = \frac{17}{30}.$$

(b) If the piece of candy chosen was a Kit-Kat, what is the probability that the box chosen by the delivery man was Uneven Break? **Use the formula for Bayes' Theorem, not a tree diagram, to answer this question.**

$$\mathbb{P}(\bar{A}|B) = \frac{\mathbb{P}(B|\bar{A})\,\mathbb{P}(\bar{A})}{\mathbb{P}(B)} = \frac{\frac{7}{10}\cdot\frac{1}{3}}{\frac{17}{30}}$$

$$\boxed{= \frac{7}{17}}$$