NAME : ______________________________

Math 60

Exam 1
April 24, 2018

Prof. Pantone

Instructions: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem.

You must show all work to receive credit.

You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	20	
2	36	
3	10	
4	14	
5	10	
6	10	
Total	100	

Section 1: True/False.

1. (20) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) If X is an independent random variable, then $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$.

True False

(b) The converse of a mathematical statement is the contrapositive of the inverse.

True False

(c) The number of ways to distribute k indistinguishable objects into n indistinguishable bins is $\binom{n+k-1}{k-1}$.

True False

(d) If $B = B_1 \cup B_2 \cup \cdots \cup B_n$, and A is an event, then

$$\mathbb{P}(A|B) = \mathbb{P}(A|B_1) + \mathbb{P}(A|B_2) + \cdots + \mathbb{P}(A|B_n).$$

True False

(e) Consider a version of the Monty Hall problem with 7 doors, hiding 4 cars and 3 goats. As before, you pick a door. Before you look behind it, Monty (intentionally) reveals a goat behind another door, and asks if you want to switch to one of the five remaining doors. True or false: Your odds of winning increase if you choose one of the five remaining doors at random.

True False

(f) The set of all finite-length binary words $\{\mathtt{0}, \mathtt{1}, \mathtt{00}, \mathtt{01}, \mathtt{10}, \mathtt{11}, \mathtt{000}, \mathtt{001}, \ldots\}$ is uncountable.

True False

(g) Chebyshev's Inequality gives an upper bound on the probability that the outcome of a random variable takes a value at least a given distance from the expected value.

True False

(h) Let $\{X_1, \ldots, X_n\}$ be a set of n random variables. If each pair (X_i, X_j) is pairwise independent (for $i \neq j$), then the set $\{X_1, \ldots, X_n\}$ is mutually independent.

True False

(i) For all positive integers n and k with $n \geq k$:

$$\binom{n}{k} = \frac{1}{\binom{k}{n}}.$$

True False

(j) The variance of a random variable X is always non-negative, even if X takes only negative values.

True False

Section 2: Shorter Answer. *You must justify all work.*

2. (36)

(a) Let S and T be events with $\mathbb{P}(S) = 0.3$ and $\mathbb{P}(T) = 0.5$. Calculate $\mathbb{P}(S \cup T \mid S \cap T)$.

(b) Let X and Y be independent random variables. Circle the equations that are always true.

$\mathbb{E}[XY + X] = \mathbb{E}[X](\mathbb{E}[Y] + 1)$

$\mathrm{Var}(XY) = \mathrm{Var}(X)\,\mathrm{Var}(Y)$

$\mathrm{Var}(X - \mathrm{Var}(X)) = 0$

$\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0$

$\mathbb{E}[\mathbb{E}[X]X] = \mathbb{E}[X^2]$

$\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

(c) Your top drawer contains 10 red socks, 6 green socks, and 4 yellow socks. If you draw two socks at random (without replacement), what is the probability that they are the same color?

(d) In a randomly shuffled deck of cards, what is the expected number of pairs of consecutive cards with equal value? (For example, in [7, 5, 5, K, 3, ...], the "5, 5" counts as one pair. In [2, A, A, A, 3, 5, ...], the "A, A, A"' counts as two consecutive pairs, and so on.)

(e) Box 1 has 2 red marbles and 3 yellow marbles, Box 2 has 4 red marbles and 2 green marbles, and Box 3 has 2 yellow marbles and 1 green marble. Suppose that you pick a box randomly and draw a marble. What is the probability that you picked Box 2 given that the marble drawn was green?

(f) Assume n and k are positive, even integers. Compute the number of ways to place n numbered (i.e., distinguishable) ping-pong balls into k distinguishable buckets such that even-numbered ping-pong balls can only go in even-numbered buckets and odd-numbered ping-pong balls can only go in odd-numbered buckets.

Section 3: Longer Answer.

If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

You must fully justify all steps!

3. (10) Suppose we have an urn containing c yellow balls and d green balls. We draw k balls, without replacement, from the urn. Find the expected number of yellow balls drawn.

4. (14) Recall the notation $[m] = \{1, 2, \ldots, m\}$. Let $f : [k] \to [n]$ be a function. Assume n and k are positive integers, but make no assumption about their relative size.

(a) How many different functions $f : [k] \to [n]$ are there?

(b) A function is said to be *injective* or *one-to-one* if everything in the domain maps to a unique element of the co-domain. Formally, $f : [k] \to [n]$ is injective if

$$\text{for all } i, j \in [k] \text{ such that } i \neq j, \text{ we must have } f(i) \neq f(j).$$

If a function is chosen at random from the set of all functions $f : [k] \to [n]$ (all equally likely to be chosen), what is the probability that it's injective?

(c) A function is said to be *surjective* or *onto* if everything in the co-domain is mapped to by something in the domain. Formally, $f : [k] \to [n]$ is surjective if

$$\text{for all } j \in [n], \text{ there exists } i \in [k] \text{ such that } f(i) = j.$$

If a function is chosen at random from the set of all functions $f : [k] \to [n]$ (all equally likely to be chosen), what is the probability that it's surjective?

Hint: Let S_i be the event that the randomly chosen function does not map anything to $i \in [n]$. Use Inclusion-Exclusion.

(d) Let Ω be the sample space of all functions $f : [k] \to [n]$. For such a function f, define the random variable $X : \Omega \to \mathbb{R}$ by $X(f) =$ [the size of the range of f] $= |\{f(i) : i \in [k]\}|$. Find $\mathbb{E}[X]$.

Hint: Your answer to part (c) *will not* be helpful.

5. (10)

(a) Let $X_1, \ldots, X_N$ be independent random variables with the same expected value and variance: $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \tau$ for all i.

Define $S_N = X_1 + \ldots + X_N$ and $A_N = S_N/N$. Find the expected value, variance, and standard deviation of S_N and A_N.

(b) Humphrey and Dwayne play a game where they flip a fair coin N times. For every heads, Humphrey wins a dollar from Dwayne, and for every tails, Dwayne wins a dollar from Humphrey. Define W_N to be Humphrey's winnings at end of the game (which can be negative!). Find the expected value, variance, and standard deviation of W_N.

6. (10) Let X and Y be independent random variables, and let $f, g : \mathbb{R} \to \mathbb{R}$ be any two functions. Prove that $f(X)$ and $g(Y)$ are independent random variables. (For example, X^7 and $\sin(Y)$ are independent – this is just an example, you need to prove it for all possible functions f and g.)

Hint: You may find it useful to define, for any function $h : \mathbb{R} \to \mathbb{R}$ and any set $S \subseteq \mathbb{R}$, the set $h^{-1}(S) = \{r \in \mathbb{R} : h(r) \in S\}$. In other words, $h^{-1}(S)$ is the set of real numbers that get mapped into S by h.