MATH 118 – HOMEWORK 4

Spring 2017

assigned Friday, May 19
due Friday, June 2

Each question is worth 10 points. The maximum possible score on this assignment is 50 points.

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1. The *total degree* of a monomial is the sum of the exponents. An ordering in which the total degree of the monomials is the most important criterion for comparison is called *graded*.

   The *grevlex* monomial ordering is defined by first choosing an ordering of the variables $\{x_1, \ldots, x_n\}$, then defining $m_1 \geq m_2$ for monomials m_1 and m_2 if either $\deg(m_1) > \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and the first exponent of $x_n, x_{n-1}, \ldots, x_1$ (in that order) where m_1 and m_2 differ is *smaller* in m_1.

   (a) Prove that grevlex is a monomial ordering that satisfies $x_1 > x_2 > \cdots > x_n$.

   (b) Prove that the grevlex ordering on $F[x_1, x_2]$ with respect to $\{x_1, x_2\}$ is the graded lexicographic ordering with $x_1 > x_2$, but that the grevlex ordering on $F[x_1, x_2, x_3]$ is not the grading of *any* lexicographic ordering. (The "grading" of an order $\geq$ is the new order $\geq_g$ formed by declaring $m_1 >_g m_2$ if $\deg(m_1) > \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and $m_1 > m_2$.)

   (c) Show that $x_1 x_2^2 x_3 > x_1^2 x_3^2 > x_2^2 x_3^2 > x_2 x_3^2 > x_1 x_2 > x_2^2 > x_1 x_3 > x_3^2 > x_1 > x_2$ for the grevlex monomial ordering with respect to $\{x_1, x_2, x_3\}$.

2. Find a Gröbner basis for the ideal $I = (x^2 + xy + y^2 - 1, x^2 + 4y^2 - 4)$ for the lexicographic ordering $x > y$ and use it to find the four points of intersection of the ellipse $x^2 + xy + y^2 = 1$ with the ellipse $x^2 + 4y^2 = 4$ in $\mathbb{R}^2$.

   (*Note:* I did not teach you the algorithm for computing Gröbner bases. The goal of this question is for you to learn how to use some Computer Algebra System (i.e., Maple, Singular, Sage, Mathematica, Macaulay2, etc) to compute them for you!)

3. Set up an unambiguous context-free grammar that models walks, meanders, and excursions on the line $\mathbb{Z}$ with step set $\{-2, -1, 0, 1, 2\}$. Use this to derive a system of polynomial equations. You do not need to solve the system.

4. Consider the Chen-grammars that we studied in class. The *Eulerian numbers* A(n,k) count many things in combinatorics, in particular $A(n,k)$ is the number of permutations in S_n with k descents. (If $\pi = \pi_1 \cdots \pi_n$ is written in one-line notation, then a descent is a consecutive pair of entries (π_i, π_{i+1}) such that $\pi_i > \pi_{i+1}$). They can be defined by the recurrence relation

   $$A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1)$$

   with initial conditions $A(0,0) = 1$ and $A(0,k) = 0$ for all $k \geq 1$. Consider the grammar $G = \{x \to xy, y \to xy\}$. Prove that

   $$D^n(x) = \sum_{k=0}^{n-1} A(n,k) x^{k+1} y^{n-k}.$$

5. Read sections 1-5 of the paper

   Symbol-crunching with the transfer-matrix method in order to count skinny physical creatures and pages 451 through the middle of 454 of

The umbral transfer-matrix method. I. Foundations,

both by Doron Zeilberger. Tell me your thoughts about Zeilberger's opinion of human vs. computer, proof vs. programming, etc. There is no right or wrong answer; I'm just looking for a thoughtful consideration of his arguments.