MATH 118 – HOMEWORK 3

Spring 2017

assigned Thursday, May 4
due Wednesday, May 17

Each question is worth 10 points. The maximum possible score on this assignment is 50 points.

Answer any combination of <u>at most</u> 5 questions.

1. Prove the following theorem that we stated in class. Let $X \subseteq \mathbb{N}^k$ be semilinear and let ψ be the Parikh map. Then, there exists a regular language $\mathcal{L} \subseteq \{a_1, a_2, \ldots, a_k\}^*$ such that $\psi(\mathcal{L}) = X$.

2. Given two words x and y over the same alphabet, we say that x is a *subword* of y if y can be written as the concatenation $y = uxv$ for some words u and v. In other words, x is a subword of y if it appears consecutively inside y.

   Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let # be a third letter not in Σ. Consider the language
   $$\mathcal{L} = \{x\#y : x, y \in \Sigma^* \text{ and } x \text{ is not a subword of } y\}.$$
   Is $\mathcal{L}$ is a context free language? Prove your answer.

3. Given two words x and y over the same alphabet, we say that x is a *subsequence* of y if some number of letters of y can be deleted to leave only x. In other words, x is a subsequence of y if it appears (not necessarily consecutively) inside y.

   Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let # be a third letter not in Σ. Consider the language
   $$\mathcal{L} = \{x\#y : x, y \in \Sigma^* \text{ and } x \text{ is not a subsequence of } y\}.$$
   Is $\mathcal{L}$ is a context free language? Prove your answer.

4. Let G be a context free grammar in Chomsky normal form that contains k variables. Show that if G generates some string with a derivation having at least 2^k steps, then the language generated by G contains an infinite number of words.

5. Construct a PDA (i.e., draw it and explain how it works) that accepts the language
   $$\mathcal{L} = \{w \in \{\mathtt{0}, \mathtt{1}\}^* : 2|w|_{\mathtt{0}} = 3|w|_{\mathtt{1}}\}.$$
   Recall that $|w|_\ell$ denotes the number of occurrences of the letter ℓ in w.

6. Show that the language
   $$\mathcal{L} = \{0^i 1^j 0^k : j = \max(i, k)\}$$
   is not context free.

7. Two words x and y over the same alphabet are rearrangements, denoted $x \sim y$, if they have the same length and the same multiset of letters. For example, $133213 \sim 112333$ but $11123 \not\sim 12333$.

   Let $\mathcal{L}$ be a regular language, and define
   $$\textsc{Rearrangements}(\mathcal{L}) = \{w : \text{there exists } z \in \mathcal{L} \text{ such that } w \sim z\}.$$
   Is $\textsc{Rearrangements}(\mathcal{L})$ regular? Is it context free? (Does it depend on the size of the alphabet of $\mathcal{L}$?) Prove your answers.
