MAS 7396 - Advanced Topics in Algebra (Lie Algebras)

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Contents

This packet consists of notes from MAS 7396 - Advanced Topics in Algebra taught during the Fall 2012 semester at the University of Florida. The course focused on the study of Lie algebras, and was taught by Prof. P. Sin.

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Chapter 1

Course Notes

1.1 Day 1 - 08/29/12

1.1.1 First Definitions

Definition: Let F be a field. A Lie algebra over F is a vector space L with a product

$$[\cdot,\cdot] : L \times L \to L$$

such that:

(L1) $[\cdot,\cdot]$ is bilinear.

(L2) $[x,x] = 0$ for all $x \in L$.

(L3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$. (This is the Jacobi Identity.)

Remark: The word "algebra" in the term "Lie algebra" has no relation to the concept of an "algebra" developed in Dummit & Foote or Hungerford. This is more of a historical use of the word.

Definition: We define a Lie ideal I to be a Lie subalgebra (i.e., a subset closed under $[\cdot,\cdot]$) such that $[x,a] \in I$ for all $x \in L$ and $a \in I$.

1.1.2 Some Examples

Examples:

(1) $\mathfrak{gl}(V) := \mathrm{End}_F(V)$, where V is an F-vector space. The operation is

$$[A,B] := AB - BA$$

where multiplication can be viewed as either composition of endomorphisms or matrix multiplication.

We now verify the axioms:

(L1) Bilinearity:

$$\begin{aligned}[A+\lambda C, B] &= (A+\lambda C)B - B(A+\lambda C)\\ &= AB + \lambda CB - BA - \lambda BC\\ &= [A,B] + \lambda[C,B].\end{aligned}$$

(L2) This is obvious since $AA - AA = 0$.

(L3) Jacobi Identity:

$$\begin{aligned}
&[A,[B,C]] + [B,[C,A]] + [C,[A,B]] \\
&\quad = A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&\quad = ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \\
&\quad = 0.
\end{aligned}$$

(2) $\mathfrak{Sl}(V) := \{A \in \mathfrak{gl}(V) \mid \mathrm{tr}(A) = 0\}$. The function $\mathrm{tr} : \mathfrak{gl}(V) \to F$ computes the trace of a matrix. Observe that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. We derive that $\dim(\mathfrak{Sl}(V)) = (\dim(V))^2 - 1$.

For example, a basis of $\mathfrak{Sl}_3$ is:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(3) $\mathfrak{Sp}(2\ell, F)$, where V is a finite dimensional F-vector space with nonsingular alternating bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

such that $\langle v, v \rangle = 0$ for all $v \in V$. By bilinearity:

$$0 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \langle v, w \rangle + \langle w, v \rangle$$

and so we deduce that

$$\langle v, w \rangle = -\langle w, v \rangle.$$

Lemma: In the above definition, V must have even dimension and additionally V has a basis $v_1, w_1, \cdots, v_n, w_n$ such that the form has the matrix

$$\left(\begin{array}{c|c|c|c}
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & 0 & 0 \\ \hline
0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 & 0 \\ \hline
0 & 0 & \ddots & 0 \\ \hline
0 & 0 & 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{array}\right).$$

Proof: We use induction on $\dim(V)$. Pick nonzero $v \in V$. Then, by nonsingularity, there exists $w \in V$ such that $\langle v, w \rangle \neq 0$. By scaling, we can assume without loss of generality that $\langle v, w \rangle = 1$. Then, let V_1 be the space spanned by v and w. So on V_1 the function $\langle \cdot, \cdot \rangle\big|_{V_1 \times V_1}$ has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence V_1 is a nonsingular subspace, and it follows that $V = V_1 \oplus V_1^\perp$, where

$$V_1^\perp := \{x \in V \mid \langle x, u\rangle = 0, \ \forall u \in V_1\}.$$

Then, $V_1^\perp$ is also nonsingular and so induction applies. □

We see that we can also write

$$\mathfrak{Sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}_{2\ell} \mid AX + XA^t = 0, \text{ for } X = \left(\begin{array}{c|c|c} 1 & & \\ \hline & & I_\ell \\ \hline & I_\ell & \end{array}\right) \right\}$$

for matrices X of a particular form which I need to check.

(4) $\mathfrak{G}(2\ell+1, F) := \left\{ A \in \mathfrak{gl}_{2\ell+1} \mid AX = XA^t = 0, \text{ for } X = \left(\begin{array}{c|c} & I_\ell \\ \hline -I_\ell & \end{array}\right) \right\}.$

(5) $\mathfrak{G}(2\ell, F) := \left\{ A \in \mathfrak{gl}_{2\ell} \mid AX = XA^t = 0, \text{ for } X = \left(\begin{array}{c|c} & I_\ell \\ \hline I_\ell & \end{array}\right) \right\}.$

The examples (2)−(5) are called types A_ℓ, C_ℓ, B_ℓ, D_ℓ, respectively.

Suggested Homework: Try to find some bases for these examples. They have a nice block form.

1.2 Day 2 - 08/31/12

1.2.1 Other Sources of Lie Algebras

Definition: An F-algebra is an F-vector space A with a bilinear multiplication operation $A \times A \to A$.

Definition: A derivation of A is a linear map $d : A \to A$ satisfying the Leibniz rule:

$$d(ab) = d(a)b + ad(b)$$

for all $a, b \in A$.

Construction: Let $\mathrm{Der}(A) \subseteq \mathrm{End}_F(A)$ be the set of all derivations of A. It is actually a subspace of $\mathrm{End}_F(A)$. Define

$$[\cdot,\cdot] : \mathrm{Der}(A) \times \mathrm{Der}(A) \to \mathrm{Der}(A)$$

by

$$[d, d'] = d \circ d' - d' \circ d$$

so that

$$[d, d'](a) = d(d'(a)) - d'(d(a)).$$

We now check that this makes $\mathrm{Der}(A)$ a Lie subalgebra of $\mathrm{End}_F(A)$.

$$\begin{aligned}
[d, d'](ab) &= d(d'(ab)) - d'(d(ab)) \\
&= d(d'(a)b + ad'(b)) - d'(d(a)b + ad(b)) \\
&= d(d'(a))b + d'(a)d(b) + d(a)d'(b) + ad(d'(b)) - d'(d(a))b - d(a)d'(b) - d'(a)d(b) - ad'(d(b)) \\
&= d(d'(a))b + ad(d'(b)) - d'(d(a))b - ad'(d(b)) \\
&= [d, d'](a)b + a[d, d'](b).
\end{aligned}$$

Remark: If A is a Lie Algebra, then for each $x \in A$, the map

$$ad_x : A \to A$$

defined by

$$ad_x(a) := [x, a]$$

is a derivation. To show this we check the Leibniz rule (we start with the rule and work toward a true statement):

$$\begin{aligned} ad_x([a,b]) &= [ad_x(a), b] + [a, ad_x(b)] \\ [x,[a,b]] &= [[x,a],b] + [a,[x,b]] \\ [x,[a,b]] - [[x,a],b] - [a,[x,b]] &= 0 \\ [x,[a,b]] + [b,[x,a]] + [a,-[x,b]] &= 0 \\ [x,[a,b]] + [b,[x,a]] + [a,[b,x]] &= 0. \end{aligned}$$

The last line is just the Jacobi identity. So, the Jacobi identity implies that ad_x is a derivation of the Lie bracket. These derivations are called inner derivations.

1.2.2 Lie Ring of a Group

Definition: Let G be a group. First recursively define the operation:

$$[x_1, x_2, \ldots, x_n] := [[x_1, \ldots, x_{n-1}], x_n].$$

We define the lower central series by

$$L_0(G) := G, \quad L_1(G) := [G, G], \quad L_2(G) := [L_1(G), G], \quad \cdots \quad L_i(G) = [L_{i-1}(G), G].$$

Note that $L_0(G) \supseteq L_1(G) \supseteq L_2(G) \supseteq \cdots$.

Lemma: Let $x, x' \in L_i(G)$, let $y, y' \in L_j(G)$, and let $z \in L_k(G)$. Then,

(i) $[L_i(G), L_j(G)] \subseteq L_{i+j+1}(G)$

(ii) $[x, y] = [y, x] \pmod{L_{i+j+1}(G)}$

(iii) $[xx', y] = [x, y][x', y] \pmod{L_{i+j+1}(G)}$ and $[x, yy'] = [x, y][x, y'] \pmod{L_{i+j+1}(G)}$

(iv) $[x, y, z][y, z, x][z, x, y] = 0 \pmod{L_{i+j+k+1}(G)}$

(v) $[x, y]^a = [x^a, y] = [x, y^a] \pmod{L_{i+j+1}(G)}$

Remark: Consider

$$L := \bigoplus_{i=0}^{\infty} L_i(G)/L_{i+1}(G).$$

Then we can define:

$$[\cdot, \cdot] : (L_i(G)/L_{i+1}G) \times (L_j(G)/L_{j+1}(G)) \to L_{i+j+1}(G)/L_{i+j+2}(G)$$

by

$$[x + L_{i+1}(G), y + L_{j+1}(G)] := [x, y] + L_{i+j+2}.$$

The properties of the **Lemma** show us that the operation is a ring. Parts (iii) and (iv) give bilinearity. Part (iv) gives the Jacobian Identity.

1.2.3 Ideals, Homomorphisms, etc.

Definition: Let L be a Lie algebra over F. A subspace $I \subseteq L$ is a Lie ideal if $[I, L] \subseteq I$.

Definition: The center is defined as

$$Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.$$

We say that L is abelian if $Z(L) = L$.

Remark: The derived subgroup

$$[L, L] := \text{span}_{\mathbb{C}}\{[x, y] \mid x, y \in L\}$$

is an ideal.

Warning: Just like with groups, the set of commutator elements is not itself an ideal!

Definition: Say L is simple if its only ideals are L and $\{0\}$.

Definition: If X is a subspace of L, the normalizer of X is

$$N_L(X) := \{y \in L \mid [y, X] \subseteq X\}.$$

This is a Lie subalgebra of L.

Definition: If X is a subspace of L, the centralizer of X is

$$C_L(X) := \{y \in L \mid [y, X] = \{0\}\}.$$

This is a Lie subalgebra of L.

Definition: If L and L' are Lie algebras over F, then a linear map $\phi : L \to L'$ is a Lie algebra homomorphism if and only if

$$[\phi(x), \phi(y)] = \phi([x, y])$$

for all $x, y \in L$.

Remark: If ϕ is a homomorphism of Lie algebras, then $\text{Ker}(\phi)$ is an ideal. Conversely, if I is an ideal, then we can construct the quotient Lie algebra L/I by defining

$$[x + I, y + I] := [x, y] + I.$$

This gives a homomorphism ψ such that $\text{Ker}(\psi) = I$ via the natural projection $L \to L/I$.

1.3 Day 3 - 09/05/12

1.3.1 Representations

Definition: A representation of a Lie algebra L is a Lie algebra homomorphism from L to $\mathfrak{gl}(V)$ for some vector space V.

The adjoint representation is the map $\text{ad} : L \to \text{Der}(L) \subseteq \mathfrak{gl}(V)$ defined by $x \mapsto \text{ad}_x$. Recall that $\text{ad}_x(y) := [x, y]$. To verify that this is a representation, we must have

$$[\text{ad}_x, \text{ad}_y] = \text{ad}_{[x,y]}\,.$$

Computing this,

$$\begin{aligned}[\text{ad}_x, \text{ad}_y](z) &= \text{ad}_x(\text{ad}_y(z)) - \text{ad}_y(\text{ad}_x(z)) \\ &= [x, [y, z]] - [y, [x, y]],\end{aligned}$$

and

$$\text{ad}_{[x,y]}(z) = [[x, y], z].$$

The equality of these two sides follows from the Jacobi identity.

We can see that the kernel of ad is the center of L.

Definition: Suppose $x \in L$ such that the derivation ad_x is a nilpotent (alternatively, "x is an ad-nilpotent") linear transformation of L. Let k be such that $(\text{ad}_x)^k = 0$ (this multiplication is composition). Define

$$\exp(\text{ad}_x) := 1 + \text{ad}_x + \frac{(\text{ad}_x)^2}{2} + \cdots + \frac{(\text{ad}_x)^{k-1}}{(k-1)!}.$$

It is clear that $\exp(\text{ad}_x) \in GL(L)$ (i.e., it's invertible). We claim that $\exp(\text{ad}_x) \in \text{Aut}(L)$, i.e.,

$$[\exp(\text{ad}_x)(y), \exp(\text{ad}_x)(z)] = \exp(\text{ad}_x)([y, z]).$$

Let δ be an arbitrary nilpotent derivation of L. Then,

$$\begin{aligned}\delta([x, y]) &= [\delta(x), y] + [x, \delta(y)] \\ \delta^2([x, y]) &= [\delta^2(x), y] + [\delta(x), \delta(y)] + [\delta(x), \delta(y)] + [x, \delta^2(y)] \\ \vdots \quad &= \quad \vdots \\ \frac{\delta^n}{n!}([x, y]) &= \sum_{i=0}^{n} \left[\frac{1}{i!}\delta^i(x), \frac{1}{(n-i)!}\delta^{n-1}(y)\right].\end{aligned}$$

This is called the *Leibniz rule*. So, assuming that $\delta^k = 0$, we have

$$\begin{aligned}[\exp(\delta)(x), \exp(\delta)(y)] &= \left[\sum_{i=0}^{k-1} \frac{\delta^i(x)}{i!}, \sum_{j=0}^{k-1} \frac{\delta^j(y)}{j!}\right] \\ &= \sum_{n=0}^{2k-2} \left(\sum_{i=0}^{n} \left[\frac{\delta^i(x)}{i!}, \frac{\delta^{n-1}(y)}{(n-i)!}\right]\right) \\ &= \sum_{n=0}^{2k-2} \frac{\delta^n([x, y])}{n!} \\ &= \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!} && \text{(since } \delta^k = 0\text{)} \\ &= \exp(\delta)([x, y]).\end{aligned}$$

Example: Let $L := SL_2$. Consider

$$x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$\begin{aligned}
[x,x] &= 0, \\
[x,h] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2x, \\
[x,y] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h.
\end{aligned}$$

So,

$$\begin{aligned}
\mathrm{ad}_x : x &\longmapsto 0 \\
h &\longmapsto -2x \\
y &\longmapsto h
\end{aligned}$$

This gives the matrix (where columns 1, 2, and 3 represent x, h, and y):

$$\mathrm{ad}_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe hence that $(\mathrm{ad}_x)^3 = 0$ and so

$$\begin{aligned}
\exp(\mathrm{ad}_x) &= 1 + \mathrm{ad}_x + \frac{(\mathrm{ad}_x)^2}{2} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Definition: The subgroup of $\mathrm{Aut}(L)$ generated by the elements $\exp(\mathrm{ad}_x)$ for ad_x nilpotent is called the subgroup of inner automorphisms. This subgroup is normal: if $\phi \in \mathrm{Aut}(L)$ then,

$$\phi(\mathrm{ad}_x)\phi^{-1} = \mathrm{ad}_{\phi(X)}$$

and so

$$\phi(\exp(\mathrm{ad}_x))\phi^{-1} = \exp(\mathrm{ad}_{\phi(x)}).$$

Remark: Just like group theory, we can define solvable and nilpotent Lie algebras.

Definition: Let L be a Lie algebra. Set $L^{(0)} := L$, $L^{(1)} := [L,L]$, $L^{(2)} := [L^{(1)}, L^{(1)}]$, ... (where we are using commutator notation). Then we say that L is solvable if $L^{(n)} = 0$ for some n.

Example: Consider the upper triangular algebra. Then, $L^{(1)}$ is the set of strictly upper triangular matrices, and $L^{(2)}$ is the set of upper triangular matrices with both the diagonal and the next diagonal above it are all zero, etc. Following this, $L^{(n-1)}$ is the set of matrices which are zero in all entries except the upper right-most element, and $L^{(n)} = 0$.

Proposition:

(a) If L is solvable then so are all homomorphic images and subalgebras.

(b) If I is a solvable ideal such that L/I is solvable, then L is solvable.

(c) If I, J are solvable ideals then $I + J$ is solvable.

Definition: The radical of L, denoted $\operatorname{Rad}(L)$, is the unique maximal solvable ideal of L (given by part (c) in the above Proposition).

Definition: If $\operatorname{Rad}(L) = 0$ then we say that L is semisimple.

Note: For any Lie algebra L, the quotient $L/\operatorname{Rad}(L)$ is semisimple.

1.4 Day 4 - 09/07/12

1.4.1 Nilpotency

Definition: Define $L^0 := L,\ L^1 := [L, L],\ \ldots,\ L^i := [L, L^{i-1}]$. This is called the lower central series. We say that L is nilpotent if $L^n = 0$ for some n.

Example: Abelian Lie algebras L are nilpotent because $L^1 = 0$.

Example: As we showed in class previously, the Lie algebra of all strictly upper triangular matrices of a particular size is nilpotent.

Proposition: Let L be a Lie algebra.

(a) If L is nilpotent then so are all subalgebras of L and all homomorphic images of L.

(b) If $L/Z(L)$ is nilpotent, then so is L.

(c) If L is nilpotent, then $Z(L) \neq 0$.

Remark: If L is nilpotent and $x \in L$ then ad_x is a nilpotent linear transformation.

Theorem: Let L be a subalgebra of $\mathfrak{gl}(V)$ with $V \neq 0$ and V finite dimensional. If L consists of nilpotent endomorphisms, then there exists a nonzero $v \in V$ such that $Lv = 0$.

Proof: We proceed by induction on $\dim(L)$. If $\dim(L) = 0$ or $\dim(L) = 1$ then the theorem is trivial. Now suppose that $K \neq L$ is a subalgebra of L. We first prove a lemma.

Lemma: If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then ad_x is nilpotent.

Proof: Write

$$\operatorname{ad}_x(y) = [x, y] = xy - yx = \lambda_x(y) - \rho_x(y).$$

So, $\lambda_x \in \operatorname{End}(\mathfrak{gl}(V))$ is left multiplication and $\rho_x \in \operatorname{End}(\mathfrak{gl}(V))$ is right multiplication.

Observe that λ_x and ρ_x are nilpotent and they commute. So, $\operatorname{ad}_x = \lambda_x - \rho_x$ is also nilpotent. $\square$

Applying the lemma to K, we see that K acts on L as an algebra of nilpotent linear transformations, hence it also asks on L/K. By induction, there exists $x+K \in L/K$ which is killed by K, i.e., $[y,x] \in K$ for all $y \in K$. Since $x \notin K$ and $x \in N_L(K)$, we have $N_L(K) > K$.

Now suppose that K is a maximal sub algebra of L. Then, we must have $N_L(K) = L$, i.e., K is an ideal of L. Also note that K has codimension 1. This is analogous to the case of p-groups, where any maximal subgroup must have index p.

So, $L = K + Fz$ for some $z \notin K$. By induction we have that the set

$$W := \{v \in V \mid Kv = 0\}$$

is nontrivial. Pick a nonzero $w \in W$. Since K is an ideal, we can prove that $LW \subseteq W$:

Suppose $w \in W$ and $x \in L$. We want to show that $xw \in W$. Let $y \in K$. We need to check that $yxw = 0$. Observe that

$$yxw = xyw - [x,y]w.$$

We have that $yw = 0$ and so $xyw = 0$. Also, $[x,y] \in K$ since K is an ideal. Hence, $[x,y]w = 0$. Therefore $yxw = 0$, which shows that $LW \subseteq W$.

So, z acts as a nilpotent endomorphism of W. Thus, there exists $v \in W$ such that $zv = 0$. Now,

$$Lv = Kv + Fzv = 0 + 0 = 0. \ \square$$

Engel's Theorem: If all elements of L are ad-nilpotent (i.e., ad_x is nilpotent for all $x \in L$), then L is nilpotent.

Proof: By the previous **Theorem**, there exists a nonzero $x \in L$ such that $(\mathrm{ad}\, L)x = 0$, i.e., $x \in Z(L)$. So, $Z(L) \neq 0$. Hence, $L/Z(L)$ consists of ad-nilpotent elements. So by induction, $L/Z(L)$ is nilpotent. Therefore L is nilpotent. $\square$

Warning: A matrix in $\mathfrak{gl}(V)$ may be *ad-nilpotent* without being *nilpotent*. For example, I.

Corollary: If L is a subalgebra of $\mathfrak{gl}(V)$ which consists of nilpotent endomorphism, then there exists a flag of L-invariant subspaces

$$0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n$$

such that

$$LV_i \subseteq V_{i-1}$$

for all i.

Proof: Observe that L acts on V. So, there exists $v \neq 0$ such that $Lv = 0$. Set $V_1 = \langle V \rangle$ and now L acts on V/V_1. $\square$

Remark: This corollary implies that there exists a basis of V such that L is a subalgebra of $n(n,F)$ (the algebra of strictly upper triangular matrices of size n.

1.5 Day 5 - 09/10/12

1.5.1 Lie's Theorem

Lie's Theorem Assume F is algebraically closed of characteristic zero. Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, for V finite dimensional. If $V \neq 0$ then V contains a common eigenvector for all endomorphisms in L.

Proof: Assume $\dim(L) \neq 0$. We proceed by induction on $\dim(L)$. Since L is solvable, we have that $[L, L] \subsetneqq L$. Any subalgebra of L containing $[L, L]$ is automatically an ideal. So, there exists an ideal $K \subseteq L$ such that $K \supset [L, L]$ and $\operatorname{codim}(K) = 1$.

By induction, since K is solvable, there exists $v \in V \setminus \{0\}$ which is a common eigenvector for K, i.e., there exists $\lambda : K \to F$ such that for all $y \in K$,

$$yv = \lambda(y)v.$$

Define

$$W := \{w \in V \mid yw = \lambda(y)w \ \ \forall y \in K\} \neq 0.$$

Note that W is a subspace of V.

We want to show that $L(W) \subseteq W$. Let $w \in W$, $x \in L$, $y \in K$. We need to test the case when $xw \in W$.

$$\begin{aligned} yxw &= yxw - xyw + xyw \\ &= [y, x]w - x\lambda(y)w \\ &= \lambda(y)xw + [y, x]w \\ &= \lambda(y)xw + \lambda([y, x])w \end{aligned}$$

So, we need $\lambda([y, x])w = 0$ to show that $xw \in W$.

Let n be the smallest possible integer such that the set

$$\{w, xw, \dots, x^n w\}$$

is linearly dependent. (Such an n exists because V is finite dimensional.) Set

$$W_0 := 0,\ W_1 := \langle w \rangle,\ \dots,\ W_i := \langle w, xw, \dots, x^{i-1}w \rangle$$

for $0 \le i \le n + 1$. So,

$$\dim(W_i) = i$$

and of course $W_{n+1} = W_n$. We now claim that y leaves each W_i invariant.

$$\begin{aligned} yx^i w &= yxx^{i-1}w \\ &= yxx^{i-1}w - xyx^{i-1}w + xyx^{i-1}w \\ &= \underbrace{[y, x]}\, x^{i-1}w_{=:\, z} + \underbrace{xyx^{i-1}w}_{\in\, x(yW_i) \subseteq xW_i \subseteq W_{i+1}} \qquad = zx^{i-1}w - x^{i-1}zw + x^{i-1}zw + xyx^{i-1}w \\ &= \underbrace{[z, x^{i-1}]w}_{\in\, K} + \underbrace{x^{i-1}zw}_{\in\, K} + xyx^{i-1}w \end{aligned}$$

So, y leaves each W_i invariant and stabilizes the chain $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n$.

Now we claim that $yx^i w = \lambda(y)x^i w \pmod{W}_i$.

$$\begin{aligned} yx^i w &= yxx^{i-1}w \\ &= yxx^{i-1}w - xyx^{i-1}w + xyx^{i-1}w \\ &= [y, x]x^{i-1}w + \lambda(y)x^i w \\ &= [y, x]x^{i-1}w - x^{i-1}[y, x]w + x^{i-1}[y, x]w + \lambda(y)x^i w \\ &= [[y, x], x^{i-1}]w + x^{i-1}[y, x]w + \lambda(y)x^i w \in W. \end{aligned}$$

So, the action of y on W_n has the matrix

$$y \sim \begin{pmatrix} \lambda(y) & & & * \\ & \lambda(y) & & \\ & & \ddots & \\ 0 & & & \lambda(y) \end{pmatrix}$$

where the i^{th} column represents W_i.

Now, $\text{tr}_{W_n}(y) = n\lambda(y)$ for all $y \in K$. In particular, it holds for elements of the form $[x, z]$ with x as above and $z \in K$. But, x and z stabilize W_n, and so $[x, z]$ acts on W_n as the commutator of two elements of $\text{End}(W_n)$, so that

$$\text{tr}_{W_i}([x, z]) = 0.$$

So,

$$n\lambda([x, z]) = 0$$

for all $z \in K$, and hence

$$\lambda([x, z]) = 0$$

for all $z \in K$, as required. □

Corollary: Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, with $\dim(V) = n < \infty$. Then, L satisfies a flag

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$$

of subspaces of V with $\dim(V_i) = i$.

Proof: Use the theorem above inductively, modding out by dimension 1 subspaces and then pulling the result back. □

Remark: Let L be solvable. If we have the adjoint representation $\text{ad} : L \to \mathfrak{gl}(L) \ni \text{ad}\, L$, then $\text{ad}\, L$ is a solvable subalgebra of $\mathfrak{gl}(L)$. So by the previous corollary, L stabilizes a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n$$

of ideals of L.

Corollary: Let L be solvable of dimension n. Then, $x \in [L, L]$ implies that $\text{ad}_L\, x$ is nilpotent. In particular, $[L, L]$ is nilpotent.

Proof: By the previous corollary, L has a basis such that $\text{ad}\, y \in t(n, F)$ (upper triangular matrices of size n) for all $y \in L$. Hence, if $x \in [L, L]$ then $\text{ad}\, x \in n(n, F)$ (strictly upper triangular matrices of size n). So, $\text{ad}\, x$ is nilpotent, and so $\text{ad}_{[L,L]}\, x$ is nilpotent. Then, by **Engel's Theorem**, $[L, L]$ is nilpotent. □

1.6 Day 6 - 9/12/12

1.6.1 Jordan-Chevalley Decomposition (additive version)

Proposition: Let F be algebraically closed and of characteristic zero. Let V be a finite dimensional vector space, and let $x \in \text{End}(V)$. Then

(a) There exist unique $x_s, x_n \in \text{End}(V)$ such that $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent, and both x_s and x_n commute with any endomorphism which commutes with x.

(b) There exist polynomials $p(T)$, $q(T)$ with zero constant term, such that $x_s = p(x)$ and $x_n = q(x)$.

(c) If $A \subseteq B \subseteq V$ are subspaces and x maps B into A, then so do x_s and x_n.

Proof: Let $a_1, \ldots, a_r$ be the distinct eigenvectors with multiplicities $m_1, \ldots, m_r$. So, the characteristic polynomial of x is

$$\prod_i (T - a_i)^{m_i} \in F[T].$$

By the Chinese Remainder Theorem, there exists $p(T)$ such that

$$p(T) \equiv a_i \mod (T - a_i)^{m_i}$$

for all i, and $p(T) \equiv 0 \mod T$.

Let $V_i := \text{Ker}((x - a_i)^{m_i})$. This is typically called the geometric eigenspace. So,

$$V = V_1 \oplus \cdots \oplus V_r.$$

On V_i, $p(x)$ acts as a scalar a_i. So, $p(x)$ is semisimple.

Let $q(T) := T - p(T)$. Then, $q(x)$ is nilpotent on each V_i because it will act on each V_i as the Jordan Canonical Form minus the diagonal part. Additionally, $p(T) \equiv 0 \mod T$ and so it has no constant term, and by definition of $q(T) = T - p(T)$ we must have that $q(T)$ has no constant term.

Now set $x_s := p(x)$ and $x_n := q(x)$. Then, $x = x_s + x_n$ and x, x_s, x_n all commute.

It remains to prove that the decomposition is unique. Suppose

$$x = x_s + x_n = y_s + y_n$$

with x_s, y_s semisimple and x_n, y_n nilpotent. Then,

$$x_s - y_s = y_n - x_n$$

and $x_s - y_s$ is semisimple and $y_n - x_n$ is nilpotent. The only way to be semisimple and nilpotent is to be 0. Hence $x_s = y_s$ and $x_n = y_n$ and so the decomposition is unique. □

1.6.2 Cartan's Criterion

Theorem: (Cartan's Criterion) Let L be a subalgebra of $\mathfrak{gl}(V)$, with $\dim(V) < \infty$. Suppose $\text{tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$. Then, L is solvable.

Remark: We first state a technical lemma from which the theorem follows.

Lemma: Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$ for V finite dimensional. Let

$$M := \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}.$$

Suppose $x \in M$ satisfies $\text{tr}(xy) = 0$ for all $y \in M$. Then, x is nilpotent.

Proof of Cartan's Criterion using the Lemma: It suffices to prove that $[L, L]$ is nilpotent. By **Engel's Theorem**, it suffices to show that every $x \in [L, L]$ is a nilpotent endomorphism (if we can show this, then by a lemma, each $x \in [L, L]$ is ad-nilpotent, and then apply **Engel's Theorem**). Now apply the lemma with $A = [L, L]$ and $B = L$. Fix $x \in [L, L]$. Then let

$$M = \{z \in \mathfrak{gl}(V) \mid [z, L] \subseteq [L, L]\}.$$

Then, $L \subseteq M$. It suffices to prove that for x there is a generator $[w,z]$ of $[L,L]$ with $w,z \in L$. Let $y \in M$.

Well,

$$\operatorname{tr}([w,z]y) = \underbrace{\operatorname{tr}(w[z,y])}_{\in [L,L]} = 0.$$

So,

$$\operatorname{tr}(xy) = 0, \ \forall x \in [L,L] \ y \in M.$$

Hence $[L,L]$ is nilpotent and so L is solvable. □

Proof of Lemma: Let $m := \dim(V)$. Let $a_1, \ldots, a_m$ be the eigenvalues of x (possibly repeated). Let $E \subseteq \mathbb{C}$ be the $\mathbb{Q}$-subspace generated by $a_1, \ldots, a_m$. It suffices to show that $E^* := \operatorname{Hom}_{\mathbb{Q}}(E, \mathbb{Q}) = 0$, which then implies that $a_1, \ldots, a_m$ were zero to begin with.

Let $f \in E^*$. We will show $f = 0$. Pick a basis of V such that x_s has a diagonal matrix. Let $y \in \mathfrak{gl}(V)$ be the element whose matrix has the diagonal

$$f(a_1), \ldots, f(a_m).$$

Let $r(T) \in F[T]$ be a polynomial with no constant term such that $r(a_i - a_j) = f(a_i) - f(a_j)$ for all $1 \leq i,j \leq m$. Since f is linear we can show that $r(T)$ is well defined (note that if $a_i - a_j = a_i' - a_j'$, then $f(a_i') - f(a_j') = f(a_i) - f(a_j)$), and in fact exists by Lagrange Interpolation.

By an earlier calculation, $\operatorname{ad} y = r(x_s)$. Note that $r(x_s)$ is also a polynomial in x with no constant term. So, $\operatorname{ad} y(B) \subseteq A$, i.e. $y \in M$. Therefore,

$$\operatorname{tr}(xy) = 0$$

by hypothesis. Since x is diagonal,

$$0 = \operatorname{tr}(xy) = \sum_{i=1}^{m} \underbrace{a_i}_{\in E} \underbrace{f(a_i)}_{\in \mathbb{Q}}.$$

Applying f to each side

$$0 = f\left(\sum_{i=1}^{m} a_i f(a_i)\right) = \sum_{i=1}^{m} f(a_i)^2.$$

Since $f(a_i) \in \mathbb{Q}$, we have $f(a_i) = 0$ for all i, and so $x = 0$. □

1.7 Day 7 - 9/14/12

1.7.1 Killing Form

Definition: Let L be a finite dimensional Lie algebra over F. For $x, y \in L$, we define the Killing form by

$$\kappa(x,y) := \operatorname{tr}_{\operatorname{End}(L)}((\operatorname{ad} x)(\operatorname{ad} y)).$$

Note that $(\operatorname{ad} x)(\operatorname{ad} y)$ should be interpreted as composition of endomorphisms. Also note that $\kappa : L \times L \to F$ is a symmetric bilinear form. This map is associative:

$$\kappa([x,y],z) = \kappa(x,[y,z]).$$

Remark: Let $S = \operatorname{Rad}\kappa = \{x \in L \mid \kappa(x,y) = 0 \ \ \forall y \in L\}$. S is an ideal of L. To see this, let $x \in S$, $y \in L$, $z \in L$, and apply associativity.

Lemma: Let I be an ideal of L. Let κ be the Killing form of L and let κ_I be the Killing form of I. Then,

$$\kappa_I = \kappa\big|_{I\times I}.$$

Proof: Let V is a vector space with a subspace W. Let $\phi \in \operatorname{End}(V)$ with $\phi(V) \subseteq W$. Then,

$$\operatorname{tr}(\phi) = \operatorname{tr}\left(\phi\big|_W\right).$$

Let $x, y \in I$. Then, $(\operatorname{ad} x)(\operatorname{ad} y) \in \operatorname{End}(L)$ and $(\operatorname{ad} x)(\operatorname{ad} y)(L) \subseteq I$. So,

$$\kappa_I = \operatorname{tr}_{\operatorname{End}(I)}((\operatorname{ad} x)(\operatorname{ad} y)) = \operatorname{tr}_{\operatorname{End}(L)}((\operatorname{ad} x)(\operatorname{ad} y)) = \kappa. \ \square$$

Example: Consider $L = \operatorname{frac} Sl(2,F)$. Let

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned} \operatorname{ad} e \ &: \ e \longmapsto 0 \\ & h \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2e \\ & f \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = h. \end{aligned}$$

Summarizing, $[e,e] = 0$, $[e,h] = -2e$, $[e,f] = h$. So,

$$\operatorname{ad} e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where the columns are the basis e, h, f. Next,

$$\begin{aligned} \operatorname{ad} h \ &: \ e \longmapsto [h,e] = 2e \\ & h \longmapsto 0 \\ & f \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2f. \end{aligned}$$

So we now know that $[h,f] = -2f$ We have the matrix

$$\operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

under the same basis. Lastly,

$$\begin{aligned} \operatorname{ad} f \ &: \ e \longmapsto [f,e] = -h \\ & h \longmapsto [f,h] = 2f \\ & f \longmapsto 0 \end{aligned}$$

Hence,

$$\operatorname{ad} f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

under the same basis.

Now we can compute the Killing form:

$$(\operatorname{ad} e)(\operatorname{ad} h) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\operatorname{ad} e)(\operatorname{ad} f) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\operatorname{ad} h)(\operatorname{ad} f) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(We only need to compute half of the entries in the Killing form because we know that it is symmetric.) So, the Killing form is

$$\kappa = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

The rows and columns have order e, h, f. Note that

$$\det(\kappa) = -128 \neq 0$$

and so κ is *nondegenerate*. Also $\operatorname{Rad} \kappa = 0$.

Theorem: Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.

Proof: Assume first that $\operatorname{Rad} L = 0$. Let $S := \operatorname{Rad} \kappa$. Then,

$$\operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y)) = 0$$

for all $x \in S$ and $y \in L$. So, by **Cartan's Criterion**, $\operatorname{ad}_L S$ is solvable and so in particular, $\operatorname{ad}_S S$ is solvable, and hence S is solvable. So, since S is an ideal, $S \subseteq \operatorname{Rad} L = 0$.

Conversely, suppose $S = 0$. Let I be an abelian ideal of L. Let $x \in I$ and $y \in L$. Then,

$$(\operatorname{ad} x)(\operatorname{ad} y) : L \to L \to I.$$

So,

$$((\operatorname{ad} x)(\operatorname{ad} y))^2 : L \to [I, I] = 0.$$

Hence, $(\operatorname{ad} x)(\operatorname{ad} y)$ is nilpotent and thus $\operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y)) = 0$. Therefore, $x \in \operatorname{Rad} \kappa$. So, $I \subseteq \operatorname{Rad} \kappa = 0$. □

Theorem: Let L be semisimple. Then, there exist ideals $L_1, \ldots, L_t$ of L which are simple Lie algebras such that

$$L = L_1 \oplus \cdots \oplus L_t.$$

Moreover, every simple ideal is *equal* to one of the L_i. Also,

$$\kappa_{L_i} = \kappa\big|_{L_i \times L_i}.$$

1.8 Day 8 - 9/17/12

1.8.1 Inner Derivations

Lemma B: Let A be an F-algebra. Then, $\mathrm{Der}(A)$ contains the semisimple and nilpotent parts of its elements.

Remark: A is just a F-vector space with a bilinear multiplication. It may be a Lie algebra, or it may not. We write multiplication in the normal way, but if A is a Lie algebra, this represents the Lie bracket operation.

Proof: Let $\delta \in \mathrm{Der}(A) \subseteq \mathrm{End}(A)$. We can write $\delta = \sigma + \nu$ where σ is semisimple and ν is nilpotent. It suffices to show that $\sigma \in \mathrm{Der}(A)$.

For $a \in F$, set

$$A_a := \{x \in A \mid (\delta - a)^k x = 0 \text{ for some } k\}.$$

Define

$$A := \bigoplus A_a$$

where the sum is taken over the eigenvalues of δ. Note that σ acts on A_a as scalar multiplication by a. Also, $A_a A_b \subseteq A_{ab}$.

Now we expand, for all $x, y \in A$:

$$(\delta - (a+b))^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - a)^{n-i}(x)(\delta - b)^i(y). \qquad (\bigstar)$$

Let $x \in A_a$ and $y \in A_b$. Then, $xy \in A_{a+b}$, and so $\sigma(xy) = (a+b)xy$. On the other hand,

$$\sigma(x)y + x\sigma(y) = axy + bxy = (a+b)xy.$$

Hence $\sigma \in \mathrm{Der}(A)$. $\square$

Theorem: If L is a semisimple Lie algebra, then $\mathrm{ad}\, L = \mathrm{Der}(L)$, i.e., every derivation of L is inner.

Proof: The homomorphism $\mathrm{ad} : L \to \mathrm{ad}\, L (\subseteq \mathrm{Der}(L))$ is an isomorphism since $\mathrm{Ker}(\mathrm{ad}) = \xi(L) = 0$.

Set $M := \mathrm{ad}\, L$ and $D := \mathrm{Der}(L)$. Then, we now show that M is an ideal of D. We now want to prove that

$$[\delta, \mathrm{ad}\, x] = \mathrm{ad}(\delta x)$$

where $\delta \in \mathrm{Der}(L)$ and the Lie bracket is taken in $\mathfrak{gl}(L)$. To do this, let $y \in L$. Now,

$$\begin{aligned}[\delta, \mathrm{ad}\, x](y) &= \delta\, \mathrm{ad}(x) y - \mathrm{ad}(x)\delta y \\ &= \delta([x,y]) - [x, \delta y] \\ &= [\delta x, y] + [x, \delta y] - [x, \delta y] \\ &= [\delta x, y] \\ &= \mathrm{ad}(\delta x)(y).\end{aligned}$$

Hence M is an ideal of D.

Now we compute the Killing form. Consider κ_D. Since Killing forms acts nicely on ideals, we have

$$\kappa_D\big|_{M \times M} = \kappa_M$$

and additionally, κ_M is nondegenerate. Then,

$$D = M \oplus M^{\perp}$$

and also $M^\perp$ is an ideal of D, by the associativity of κ_D. Therefore, $[M, M^\perp] \subseteq M \cap M^\perp = 0$. So, M and $M^\perp$ commute.

Therefore, $\operatorname{ad}(\delta x) = 0$ for all $\delta \in M^\perp$ and $x \in L$. Hence, $\delta x = 0$ for all $\delta \in M^\perp$ and $x \in L$, and so $\delta = 0$, i.e., $M^\perp = 0$. □

1.8.2 Abstract Jordan Decomposition

Let L be a semisimple Lie algebra. Then, the map

$$L \to \operatorname{ad} L = \operatorname{Der} L$$

is an isomprhism.

For $x \in L$, let x_s and x_n be defined by

$$\operatorname{ad} x = \underbrace{\operatorname{ad}(x_s)}_{=(\operatorname{ad} x)_s} + \underbrace{\operatorname{ad}(x_n)}_{=(\operatorname{ad} x)_n}.$$

1.8.3 Representations and Modules

Definition: Let L be a Lie algebra. A representation of L is a homomorphism of Lie algebras:

$$\rho : L \to \mathfrak{gl}(V)$$

for some F-vector space V. Recall that we must have

$$\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x)$$

by the definition of Lie algebra homomorphism.

Definition: An L-module is a vector space V together with a map $L \times V \to V$ defined by $(x, v) \mapsto x \cdot v$, such that

(M1) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$

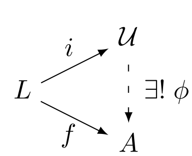
(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$

(M3) $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$

for all $x, y \in L$, $v, w \in V$, $a, b \in F$.

1.8.4 Universal Enveloping Algebra

Definition: Let L be a Lie algebra over F. A universal enveloping algebra of L is a pair $(\mathcal{U}, i)$ where $\mathcal{U}$ is an associative algebra with 1 and $i : L \to U$ is a linear map satisfying $i([x, y]) = i(x)i(y) - i(y)i(x)$ for all $x, y \in L$ and satisfying the following universal property: For any associative algebra A with 1 and any linear map $f : L \to A$ such that $f([x, y]) = f(x)f(y) - f(y)f(x)$, there exists a unique homomorphism ϕ making the following diagram commutes:

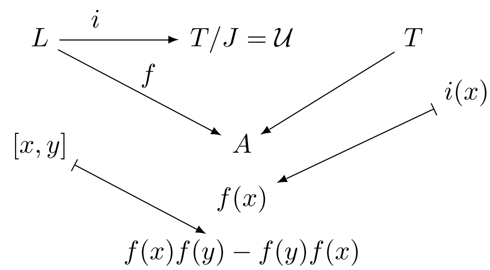


Proof of Existence: Let T be the tensor algebra on L, i.e.,

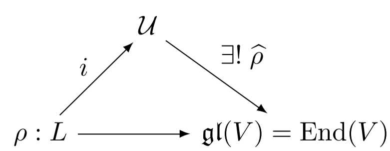
$$T = F \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \cdots = \bigoplus_{n=0}^{\infty} L^{\otimes n}.$$

Denote T^k to be the term $L^{\otimes k}$. Let J be an ideal of T generated by $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in L = T^1$.

Let $\mathcal{U} := T/J$ and let $i : L \to \mathcal{U}$. Then, $\mathcal{U}$ is an associative algebra with 1. Then,



We see that



1.9 Day 9 - 9/19/12

1.9.1 Representation Definitions

Definition: A representation (or module) L is irreducible (simple) if the only L-submodules are 0 and the whole module.

Definition: We say that a representation (or module) is completely reducible (semisimple) if it is the direct sum of irreducible (simple) representations (modules).

Remark: We define the direct sum of representatoins by taking block matrices and adjoining them along a diagonal.

Definition: If V is an L-module, we define

$$\text{End}_L(V) := \{\phi \in \text{End}(V) \mid \phi(xv) = x\phi(v),\ \forall x \in L, v \in V\}.$$

Schur's Lemma: If V is irreducible, then $\text{End}_L(V) = \{F 1_V\}$ consists of scalar multiplication.

1.9.2 Module Constructions

Dual / Contragredient Representation: Let V be an L-module and let $V^* := \mathrm{Hom}(V, F)$ be the dual vector space. Define a map $L \times V^* \to V^*$ by

$$(xf)(v) \mapsto -f(xv).$$

To show that we get an L-module, we would need to show that

$$([x, y]f)(v) = x(yf)(v) - y(xf)(v).$$

Given a group G, the map $G \to GL(V)$ definds the $\mathbb{F}G$-modules with

$$(gf)(v) = f(g^{-1}(v)).$$

Tensor Product Construction: Let V and W be L-modules (where L is a Lie algebra). Make $V \otimes_F W$ into an L-module by

$$x(v \otimes w) := xv \otimes w + v \otimes xw$$

(as usual, defined by its action on the generating set of simple tensors). We would need to check that:

$$[x, y](v \otimes w) = xy(v \otimes w) - yx(v \otimes w)$$

If we let V and W be $\mathbb{F}G$-modules, then $V \otimes_F W$ is an $\mathbb{F}G$-module by the construction

$$g(v \otimes w) = gv \otimes gw.$$

To check this, we verify that

$$\begin{aligned}(hg)(v \otimes w) &= (hg)v \otimes (hg)w \\ &= h(gv) \otimes h(gw) \\ &= h(gv \otimes gw) \\ &= h(g(v \otimes w)).\end{aligned}$$

Hom Construction: Let V and W be L-modules. We can make $\mathrm{Hom}_F(V, W)$ an L-module by

$$(x\phi)(v) := x(\phi(v)) - \phi(xv).$$

Similarly, let V and W be $\mathbb{F}G$-modules. We can make $\mathrm{Hom}_F(V, W)$ into an $\mathbb{F}G$-module by

$$(g\phi)(v) := g(\phi(g^{-1}v)).$$

Remark: Consider the map $V^* \otimes W \to \mathrm{Hom}_F(V, W)$ defined by $f \otimes w \mapsto \phi$ such that

$$\phi : v \mapsto f(v)w.$$

This map is surjective, and additionally it is an isomoprhism of L-modules if $\dim(V)$ and $\dim(W)$ are finite.

1.10 Day 10 - 9/21/12

1.10.1 Weyl's Theorem

Theorem: (Weyl) Let $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra. Then, ϕ is completely reducible.

Remark: Before we can prove Weyl's Theorem, we will need to set up some machinery.

Definition: Let $\phi : L \to \mathfrak{gl}(V)$ be a faithful (i.e., injective) representation. Define an associative bilinear form on L by

$$\beta(x, y) = \mathrm{tr}_V(\phi(x)\phi(y)).$$

This is nondegenerate because

$$\mathrm{Rad}(\beta) \subseteq \mathrm{Rad}(\phi(L)) = 0$$

using **Cartan's Criterion**. Let $\{x_1, \dots, x_n\}$ be a basis of L (with $n := \dim(L)$) and let $\{y_1, \dots, y_n\}$ be the dual basis with respect to β, i.e., $\beta(x_i, y_j) = \delta_{i,j}$. Next define

$$c_\phi = \sum_{i=1}^{n} \phi(x_i)\phi(y_i) \in \mathrm{End}(V).$$

c_ϕ is called the **Casamir Element** of the representation ϕ.

Claim: $c_\phi \in \mathrm{End}_L(C)$.

Proof: Let $x \in L$. Write

$$[x, x_i] = \sum_{j=1}^{n} a_{i,j} x_j$$

and

$$[x, y_i] = \sum_{j=1}^{n} b_{i,j} y_i.$$

Now,

$$\begin{aligned} a_{i,j} &= \beta([x, x_i], y_j) \\ &= \beta(-[x_i, x], y_j) \\ &= \beta(-x_i, [x, y_j]) \\ &= -b_{j,i}. \end{aligned}$$

Recall the identity:

$$[x, yz] = [x, y]z + y[x, z]$$

in $\mathrm{End}(V)$. So, applying this to each term in the sum of c_ϕ:

$$\begin{aligned} [\phi(x), c_\phi] &= \sum_{i=1}^{n} [\phi(x), \phi(x_i)]\phi(y_i) + \sum_{i=1}^{n} \phi(x_i)[\phi(x), \phi(y_i)] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\phi(x_j)\phi(y_i) + \sum_{i=1}^{n} \phi(x_i) \sum_{j=1}^{n} b_{i,j}\phi(y_j) \\ &= 0 \end{aligned}$$

where the terms in the left sum cancel with the terms in the right sum by the fact that $a_{i,j} = -b_{j,i}$. Since the commutator is 0, $\phi(x)$ commutes with c_ϕ for all x, which proves the theorem. □

Remark: Let's compute $\mathrm{tr}(c_\phi)$:

$$\begin{aligned} \mathrm{tr}(c_\phi) &= \sum_{i=1}^{n} \mathrm{tr}(\phi(x_i), \phi(y_i)) \\ &= \sum_{i=1}^{n} \beta(x_i, y_i) \\ &= n = \dim(L). \end{aligned}$$

If V is irreducible, then

$$c_\phi = \frac{\dim(L)}{\dim(V)} \cdot I_V.$$

Example: $L = \mathfrak{Sl}(2, \mathfrak{F})$ and $V = F^2$. Consider the standard basis e, h, f of L and the dual basis (which after some calculation, we find is $f, \frac{h}{2}, e$. Now,

$$\beta(x, y) = \mathrm{tr}(xy)$$

in this case. So,

$$c_\phi = ef + \frac{1}{2}h^2 + fe = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

Lemma: If $\phi : L \to \mathfrak{gl}(V)$ as a representation of a semisimple Lie algebra, then $\phi(L) \subseteq \mathfrak{Sl}(V)$. In particular, L acts trivially on any 1-dimensional L-module.

Proof of Weyl's Theorem: We want to show that every short exact sequence of L-modules of the form

$$0 \longrightarrow W \longrightarrow V \longrightarrow U \longrightarrow 0 \qquad (\bigstar)$$

splits. We look at three cases:

1) $0 \longrightarrow W \longrightarrow V \longrightarrow F \longrightarrow 0$, where W is simple.
2) $0 \longrightarrow W \longrightarrow V \longrightarrow F \longrightarrow 0$, where W is arbitrary.
3) The general case in ($\bigstar$).

Case 1: We can assume that L acts faithfully on V. Let c_ϕ be the Casimir element of $\phi : L \to V$. L acts trivially on F and so

$$c_\phi = \sum_{i=1}^{n} \phi(x_i)\phi(y_i)$$

maps V into W. So c_ϕ has trace 0 on $V/W \cong F$. Also, c_ϕ acts as scalars on W. This scalar is nonzero since

$$\mathrm{tr}_V(c_\phi) = \frac{\dim(L)}{\dim(V)} \neq 0.$$

Therefore, $\mathrm{Ker}(\phi)$ is 1-dimensional and intersects W trivially, i.e.,

$$V = \mathrm{Ker}(c_\phi) \oplus W$$

as L-modules. It follows directly that the short exact sequence splits.

Case 2: Proceed by induction on $\dim(W)$. Let $W' \subseteq W$ be a maximal submodule. Consider

$$0 \longrightarrow W/W' \longrightarrow V/W' \longrightarrow F \longrightarrow 0$$

which splits by **Case 1**. So, there exists $\widetilde{X}$ such that $\widetilde{X} \subseteq V$, $\widetilde{X} \supseteq W'$ and $\widetilde{X}/W' \cong F$. So, as L-modules

$$V/W' = W/W' \oplus \widetilde{X}/W'.$$

Therefore, we have a short exact sequence

$$0 \longrightarrow W' \longrightarrow \widetilde{X} \longrightarrow \widetilde{X}/W' \cong F \longrightarrow 0.$$

Since $\dim(W') < \dim(W)$, induction applies. So,

$$\widetilde{X} = W' \oplus X$$

where $X \cong F$. Check now that $V = W \oplus X$. It follows directly that the short exact sequence splits.

Case 3: We need to find a splitting map in $\mathrm{Hom}_L(V,W)$. In $\mathrm{Hom}(V,W)$ define

$$\mathcal{V} := \{f \in \mathrm{Hom}(V,W) \mid f\big|_W \text{ is scalar}\},$$

$$\mathcal{W} := \{f \in \mathrm{Hom}(V,W) \mid f\big|_W = 0\}.$$

We claim that $\mathcal{V}, \mathcal{W}$ are L-submodules of $\mathrm{Hom}(V,W)$. (The module structure is $(xg)(v) = xg(v) - g(xv)$. Check this on your own.) Then we have $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{V}/\mathcal{W} \cong F$. Thus we have the short exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{V} \longrightarrow F \longrightarrow 0.$$

By **Case 2**, $\mathcal{V}$ has a 1-dimensional submodule $\mathfrak{X}$ on which L acts trivially. Choose $f \in \mathfrak{X}$ such that $f\big|_W = 1_W$. Then, $f \in \mathrm{Hom}_L(V,W)$ since L acts trivially on f. So, f is the required splitting map. This completes this case.

So, the theorem is proved. □

1.11 Day 11 - 9/24/12

1.11.1 Preservation of Jordan Decomposition

Definition: Let $x \in L$ where L is a semisimple Lie algebra. The abstract Jordan decomposition is $x = s + n$ where

$$\mathrm{ad}\, x = \underbrace{(\mathrm{ad}\, x)_s}_{=\mathrm{ad}(s)} + \underbrace{(\mathrm{ad}\, x)_n}_{=\mathrm{ad}(n)}.$$

Theorem: If $L \subseteq \mathfrak{gl}(V)$ is a semisimple Lie algebra (with V finite dimensional), then the semisimple and nilpotent parts of each element of L lie in L. In particular, the abstract and usual Jordan Decomposition are the same.

Proof: Let $x \in L \subseteq \mathfrak{gl}(V)$. Write $x = x_s + x_n$ under the ordinary Jordan decomposition. It suffices to show that $x_s \in L$.

In the next part, "ad"="$\mathrm{ad}_{\mathfrak{gl}(V)}$". We know that

$$(\mathrm{ad}\, x)(L) \subseteq L$$

and therefore,

$$(\mathrm{ad}\, x)_s(L) \subseteq L$$

where $(\mathrm{ad}\, x)_s = \mathrm{ad}\, x_s$ by **Lemma A**. So,

$$(\mathrm{ad}\, x)_n(L) \subseteq L$$

where also $(\mathrm{ad}\, x)_n = \mathrm{ad}\, x_n$. Hence,

$$(\mathrm{ad}\, x_s)(L) \subseteq L$$

and

$$(\mathrm{ad}\, x_n)(L) \subseteq L,$$

i.e.,

$$x_s, x_n \in N_{\mathfrak{gl}(V)}(L) =: N.$$

For any L-submodule W of V, set

$$L_W := \left\{y \in \mathfrak{gl}(V) \mid y(W) \subseteq W \text{ and } \mathrm{tr}\left(y\big|_W\right) = 0\right\}.$$

Certainly $L \subseteq L_W$ for all W since $L = [L, L] \subseteq \ker(\mathrm{tr}_W)$. Define

$$L' := \bigcap_W L_W.$$

So, $L \subseteq L'$. We want $L = L'$. Now, L' is an L-submodule of $\mathfrak{gl}(V)$ containing L. Hence, by **Weyl's Theorem**, $L' = L \oplus M$ as L-modules, for some L-module M. It remains to show that $M = 0$.

Let $y \in M$. Consider that $[L, y] = 0$ since $y \in N$ and M is an L-submodule. Let W be any simple L-submodule of V. Then, $y\big|_W \in \mathrm{End}_L(W)$ and so $y\big|_W$ is a scalar. But, by definition of L', $\mathrm{tr}\left(y\big|_W\right) = 0$. Hence $y\big|_W = 0$. Since V is a direct sum of simple submodules, we get $y = 0$. Therefore $M = 0$ and so $L = L'$ and the proof is complete. □

Corollary: If L is semisimple and $\phi : L \to \mathfrak{gl}(V)$ is a finite dimensional representation, then if $x = s + n$ is the abstract Jordan decomposition of $x \in L$, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$ in $\mathfrak{gl}(V)$.

Proof: L has a basis of eigenvalues of $\mathrm{ad}\, s$. Hence $\phi(L)$ has a basis of eigenvalues of $\mathrm{ad}_{\phi(L)}(\phi(s))$. Similarly, $\mathrm{ad}_{\phi(L)}\,\phi(n)$ is nilpotent and so commutes with $\mathrm{ad}_{\phi(L)}\,\phi(s)$. Hence,

$$\phi(x) = \phi(s) + \phi(n)$$

where $\phi(s)$ is $\mathrm{ad}_{\phi(L)}$ semisimple and $\phi(n)$ is $\mathrm{ad}_{\phi(L)}$ nilpotent.

So, this is the abstract Jordan decomposition for $\phi(L)$. Hence by the **Theorem**, $\phi(s) = \phi(x)_s$ and $\phi(n) = \phi(x)_n$. □

Example: Simple modules for $\mathfrak{Sl}(2, F) =: L$. Recall the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = g, \quad [e, e] = [h, h] = [f, f] = 0.$$

To study these, we make a definition and prove some statements:

Definition: Let V be a finite dimensional L-module. Then, h acts semisimply on V (because of the persistence of the Jordan decomposition – if it acts faithfully on one representation then it acts faithfully on all representations). So,

$$V = \bigoplus_{\lambda \in F} V_\lambda$$

where V_λ is the eigenspace of V corresponding to λ, i.e., $V_\lambda = \{v \in B \mid hv = \lambda v\}$. If $V_\lambda \neq 0$, we say that λ is a <u>weight</u> of h in V, and that V_λ is a <u>weight space</u>.

Lemma: If $v \in V_\lambda$, then $ev \in V_{\lambda+2}$ and $fv \in V_{\lambda-2}$.

Proof:

$$\begin{aligned} hev &= (he - eh)v + ehv \\ &= [h, e]v + \lambda ev \qquad\qquad = 2ev + \lambda ev \\ &= (\lambda + 2)ev. \end{aligned}$$

Therefore, $ev \in V_{\lambda+2}$. The other calculation is analogous. □

Definition: Let $\dim(V) < \infty$ and write $V = \bigoplus V_\lambda$. Then, there exists λ such that $V_\lambda \neq 0$ and $V_{\lambda+2} = 0$. For such a λ, if $v \in V_\lambda$, we have $ev = 0$. A nonzero vector of such a V_λ is called a maximal vector.

Definition: Assume that V is a simple L-module. Let $v_0 \in V_\lambda$ be a maximal vector. Set

$$v_{-1} := 0, \quad v_i := \left(\frac{1}{i!}\right) f^i v_o$$

for $i \geq 0$.

Lemma: We clearly have that:

(a) $hv_i = (\lambda - 2i)v_i$

(b) $fv_i = (i+1)v_{i+1}$

(c) $ev_i = (\lambda - i + 1)v_{i-1}$

Proof of (c): Use induction on i. The base case $i = 0$ is trivial. Next consider

$$\begin{aligned} iev_i &= efv_{i-1} && \text{(by part (b))} \\ &= (ef - fe)v_{i-1} + fev_{i-1} \\ &= hv_{i-1} + (\lambda - i + 2)fv_{i-2} && \text{(second summand by induction)} \\ &= (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)(i-1)v_{i-1} \\ &= i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

This completes the inductive step.

1.12 Day 12 - 9/26/12

1.12.1 Construction of Simple Irreducible Modules

Theorem: Let V be a finite dimensional irreducible module for $L = \mathfrak{Sl}(2, F)$. Then, the following is true.

(a) V is the direct sum of weight spaces (relative to h) V_μ, where $\mu = m, m-2, \ldots, -m$ where $\dim(V) = m+1$ and $\dim(V_\mu) = 1$ for each μ.

(b) V has (up to scalars) a unique maximal vector (whose weight is m).

(c) The action of L on V is as previously given. In particular, there is at most one simple module for each m.

Construction of simple modules:

We can show existence of the simple module $V(m)$ of dimension $m+1$ for all $m = 0, 1, \ldots$. Consider the represention as a linear map

$$\phi : L \to GL(V)$$

where for all $x, y \in L$ we want to check

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

Since $[\cdot, \cdot]$ is bilinear, it suffices to check this for x, y belonging to some fixed basis.

Let $V(m)$ be an $(m+1)$-dimensional vector space with basis $v_0, \ldots, v_m$. Define, $E, H, F \in \mathrm{End}(V(m))$ by

$$\begin{aligned} Hv_i &= (m-2i)v_i \\ Fv_i &= (i+1)v_{i+1} \\ Ev_i &= (m-i+1)v_{i-1}, \quad i \geq 0 \\ Ev_0 &= 0. \end{aligned}$$

It needs to be checked that the map $\mathfrak{Sl}(2,F) \to \mathfrak{gl}(V(m))$ defined by $e \mapsto E$, $f \mapsto F$, and $h \mapsto H$ is a homomorphism of Lie algebras.

Another construction:

Let $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be a basis for F^2, under the standard representation of $\mathfrak{Sl}(2,F)$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $eX = 0$, $eY = X$, $hX = X$, $hY = -Y$, $fX = Y$ and $fY = 0$.

Consider the polynomial ring $F[X,Y]$ and let $z \in \mathfrak{gl}Sl(2,F)$ act by the derivation rule

$$z(fg) = (zf)g + f(zg).$$

It should be checked that this action is well-defined.

Then the space of homogenous polynomials of degree m is a submodule of dimension $m+1$, and irreducible. Observe for example that

$$e(X^2) = eX \cdot X + X \cdot eX = 0$$

$$f(X^2) = fX \cdot X + X \cdot fX = 2XY$$

$$f(XY) = Y^2 + X \cdot f(Y) = Y^2.$$

What does it mean in this context to say "X^2" when X is a column vector? We are thinking of X and Y as variables in the polynomial ring, so we consider the tensor algebra

$$T(V) = F \oplus V \oplus (V \otimes V) + \cdots$$

and the symmetric algebra

$$S(V) = T(V)/\langle x \otimes y - y \times x \rangle$$

and the operations are being performed in the symmetric algebra.

1.13 Day 13 - 9/28/12

1.13.1 Root Space Decomposition / Maximal Toral Subalgebras

Definition: A subalgebra of a semisimple Lie algebra L is toral if it consists of semisimple elements.

Example: The subalgebra of diagonal matrices in $\mathfrak{Sl}(V)$ is toral.

Lemma: A toral subalgebra is abelian.

Proof: Let T be toral. We need to show that $\operatorname{ad}_T x = 0$ for all $x \in T$. Suppose toward a contradiction that there exists $y \neq 0$ and $a \neq 0$ such that

$$[x, y] = ay.$$

Then,

$$(\operatorname{ad}_T y)(x) = [y, x] = -ay$$

So, $-ay$ is an eigenvector for $\operatorname{ad}_T y$ with eigenvalue 0. Write

$$x = v_1 + \cdots + v_r$$

where the v_i are the eigenvalues of $\operatorname{ad}_T y$. So,

$$(\operatorname{ad}_T y)(x) = \lambda_1 v_1 + \cdots + \lambda_k v_k$$

for $\lambda_i \neq 0$. This is a contradiction. □

Remark: Let H be a maximal toral subalgebra, i.e., not contained in any other toral subalgebra. (For example, in $\mathfrak{Sl}(n, F)$, the digonal subalgebra is a maximal toral subalgebra: pick a matrix with distinct eigenvectors along the diagonal, and now observe that any matrix that commutes with this matrix must also be diagonal.) Then, H is abelian, and so $\operatorname{ad}_L H$ is a set of commuting diagonalizable endomorphisms. So, L has a basis over which all $\operatorname{ad}_L h$ for $h \in H$ are diagonal. Each element v of such a basis defines a function $\lambda : H \to F$ by $(\operatorname{ad}_L h)(v) := \lambda(h)v$. λ is a Lie algebra homomorphism (an element of H^*), and is called a root.

Remark: We can write

$$L = \bigoplus_{\alpha \in H^*} L_\alpha$$

where

$$L(\alpha) = \{x \in L \mid (\operatorname{ad}_L h)(x) = \alpha(h)x\}.$$

This is called the root space decomposition. Accordingly, we can write

$$\mathfrak{Sl}(2, F) = Fe \oplus \underbrace{Fh} = H \oplus Ff$$

with

$$\alpha_e = 2 \qquad \alpha_h = 0 \qquad \alpha_f = -2.$$

Definition: The set $\Phi = \{\alpha \mid \alpha \neq 0,\ L_\alpha \neq 0\}$ is called the set of roots.

Proposition: For all $\alpha, \beta \in H^*$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

Proof: Let $h \in H$, $x \in L_\alpha$, $y \in L_\beta$. Then,

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha([x, y]) + \beta([x, y]) = (\alpha + \beta)([x, y]).\ \square$$

Lemma: If $\alpha + \beta \neq 0$, then L_α and L_β are orthogonal.

Proof: Pick $h \in H$ such that $(\alpha + \beta)(h) \neq 0$. Let $x \in L_\alpha$ and $y \in L_\beta$. Then,

$$\begin{aligned}\alpha(h)K(x, y) &= K([h, x], y) \\ &= -K([x, h], u) \\ &= -K(x, [h, y]) \\ &= -\beta(h)K(x, y).\end{aligned}$$

So, $(\alpha + \beta)(h)K(x, y) = 0$ and hence $K(x, y)$. This completes the theorem. □

Corollary: L_0 is orthogonal to L_α for all $\alpha \neq 0$. So, $K\big|_{L_0 \times L_0}$ is nondegenerate.

Lemma: If x, y are commuting endomorphisms and y is nilpotent, then so is xy. In particular, $\operatorname{tr}(xy) = 0$.

Theorem: $C_L(H) = H$

Proof: Set $C := C_L(H)$.

(1) If $x \in C$, then $x_n \in C$ and $x_s \in C$.

(2) All semisimple elements of C lie in H (by the maximality of H as a toral subalgebra).

(3) $K\big|_H$ is nondegenerate. Suppose $h \in H$ with $K(h, H) = 0$. Let $x \in C$ and $x = x_s + x_n$ with $x_s, x_n \in C$. Then, $x_s \in H$, so $K(h, x_s) = 0$. Also, $\operatorname{ad} x_n$ is nilpotent and commutes with $\operatorname{ad} h$. So, $\operatorname{tr}(\operatorname{ad} x_n \operatorname{ad} h) = K(x_n, h) = 0$. Hence, $K(h, x) = 0$ for all $x \in C$. Thus $h = 0$.

(4) C is nilpotent. (This means as a Lie algebra, which is a different sense of nilpotency than elements!) Let $x \in C$ and write $x = x_s + x_n$ with $x_s, x_n \in C$. Then $x_s \in H$ and so $\operatorname{ad}_C x_s = 0$. $\operatorname{ad}_L x_n$ is nilpotent and so $\operatorname{ad}_C x_n$ is nilpotent. So, $\operatorname{ad}_C x$ is nilpotent for all $x \in C$ since $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ commute. By **Engel's Theorem**, C is nilpotent.

(5) $H \cap [C, C] = 0$. Observe that

$$K(H, [C, C]) = K(\underbrace{[H, C]}_{=0}, C) = 0.$$

Since $K\big|_{H \times H}$ is nondegenerate, $H \cap [C, C] = 0$.

(6) C is abelian. Suppose otherwise that $[C, C] \neq 0$. Then, $Z(C) \cap [C, C] \neq 0$. Let $z \in Z(C)$ be nonzero. The, $z \notin H$ and so $z_n \neq 0$. Note $z_n \in Z(C)$. For any $y \in C$, $K(z_n, y) = \operatorname{tr}(\operatorname{ad} z_n \operatorname{ad} y) = 0$. So $z_n = 0$, which is a contradiction.

(7) $C = H$. If not, then there exists a nilpotent element x in C. Then, for all $y \in C$, $K(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$, since C is abelian. This is a contradiction since $K\big|_{C \times C}$ is nondegenerate.

This completes the theorem. □

Remark: We can now complete the decomposition:

$$L = \underbrace{H}_{=L_0} \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Corollary: $K\big|_{H \times H}$ is nondegenerate, so we can use K to identity H with H^*, i.e.,

$$\forall \alpha \in H^*,\ \exists! t_\alpha \in H \text{ such that } \forall h \in H : \alpha(h) = K(t_\alpha, h).$$

Look at $\{t_\alpha \mid \alpha \in \Phi\} \subseteq H$.

Proposition: Proofs of each parts are quick sketches.

(a) Φ spans H^*.

Proof: This is equivalent to saying that $\{t_\alpha\}_{\alpha \in \Phi}$ span H. If not, then there exists a nonzero $h \in H$ orthogonal to all t_α, i.e., $\alpha(h) = 0$ for all $\alpha \in \Phi$. Then, $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$. But $[h, L_0] = 0$, so $h \in Z(L) = 0$, which is a contradiction. □

(b) If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

Proof: Recall and earlier propsition: if $\alpha+\beta \neq 0$ then $K(L_\alpha, L_\beta) = 0$. Also, $L_\alpha, L_\beta \subseteq L_{\alpha+\beta}$. So the elements of L_α are nilpotent. Hence, $K(L_\alpha, L_\alpha) = 0$. If $-\alpha \notin \Phi$, then the Killing form of L_α with everything is zero, which is a contradition. □

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$ and $y \in L_{-\alpha}$. Then, $[x, y] = K(x, y)t_\alpha$.

Proof: Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$, and $h \in H$. Then,

$$\begin{aligned}\kappa(h, [x, y]) &= \kappa([h, x], y)\\ &= \alpha(h)\kappa(x, y)\\ &= \kappa(t_\alpha, h)\kappa(x, y)\\ &= \kappa(\kappa(x, y)t_\alpha, h)\\ &= \kappa(h, \kappa(x, y)t_\alpha)\end{aligned}$$

So, h is orthogonal to $[x, y] - \kappa(x, y)t_\alpha$. □

(d) If $\alpha \in \Phi$, then $[L_\alpha, L_{-\alpha}]$ is 1-dimensional with basis t_α.

Proof: Use part (c). □

(e) $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$ for $\alpha \in \Phi$.

Proof: The assertion that needs to be proved is $\kappa(t_\alpha, t_\alpha) \neq 0$. Suppose this is false. Then, $[t_\alpha, L_\alpha] = 0$ and $[t_\alpha, L_{-\alpha}] = 0$. For $x \in L_\alpha$, pick $y \in L_{-\alpha}$, with $[x, y] = t_\alpha$. Then, $\langle x, y, t_\alpha\rangle$ is solvable (because its derived subalgebra is just the span of t_α and so the second commutator is zero).

(f) If $\alpha \in \Phi$ and $0 \neq x_\alpha \in L_\alpha$ then there exists $y_\alpha \in L_{-\alpha}$ such that x_α, y_α and $h_\alpha := [x_\alpha, y_\alpha]$ span a 3-dimensional subalgebra of L isomorphism to $\mathfrak{Sl}(2, F)$:

$$x_a \mapsto \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix} \qquad y_\alpha \mapsto \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix} \qquad h_\alpha \mapsto \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

(g) $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$, $h_\alpha = -h_{-\alpha}$.

1.14 Day 14 - 10/01/12

1.14.1 Integrality Properties

Consider $\mathfrak{Sl}(2, F)$ and a finite dimensional module V. Then,

$$V \cong \bigoplus_{m \geq 0} V(m)^{(e_m)}$$

where $V(m)$ is simple of dimension $m + 1$. Observe that

$$\sum_m e_m = \dim(V_0) + \dim(V_1)$$

where $V_0 = \{v \in V \mid hv = 0\}$ and $V_1 = \{v \in V \mid hv = v\}$.

Fix $\alpha \in \Phi$ so that $-\alpha \in \Phi$. Let $S_\alpha \cong \mathfrak{Sl}(2, F)$ be a subalgebra given by $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ and $h_\alpha = [x_\alpha, y_\alpha]$. Consider the action of S_α on L by ad.

Let M be the subspace of L spanned by H and all root spaces of the form $L_{c\alpha}$ where $c \in F^\times$. M is an S_α-submodule of L. Recall $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

We now claim that $M = H + S_\alpha$, i.e., $M = H \oplus Fx_\alpha \oplus Fy_\alpha$. This will imply that

(1) The only multiples of α in Φ are $\pm\alpha$.

(2) $L_\alpha = Fx_\alpha$ is one-dimensional.

Consider the weights of h_α on M. If $x \in L_{2\alpha}$ then $[h_\alpha, x] = (c\alpha)(h_\alpha)x = (2c)x$, so we know that $2c$ is an integers, i.e. that c is an integer multiple of $1/2$.

Now consider $\mathrm{Ker}(\alpha) = \{h \in H \mid \alpha(h) = 0\}$. α is nonzero and so the kernel has codimension 1 in H. We can write $H = \mathrm{Ker}(\alpha) \oplus Fh_\alpha$ as vector spaces. So,

$$M = H \oplus \bigoplus_{c \neq 0} L_{c\alpha}$$

with $H = \{m \in M \mid [h_\alpha, m] = 0\}$. So, $M_0 = H$.

$\mathrm{Ker}(\alpha)$ is an S_α-submodule of M. If $h' \in \mathrm{Ker}(\alpha)$, then

$$[h', x_\alpha] = 0,$$
$$[h', y_\alpha] = 0,$$
$$[h', h_\alpha] = 0.$$

Hence the S_α-submodule $\mathrm{Ker}(\alpha)+S_\alpha$ contains M_0. So, $\mathrm{Ker}(\alpha)+S_\alpha$ contains the sum of all simple submodules of L considered as S_α=modules which has even highest weights, i.e., all those which contain even weight.

Hence the only possible even weights for h_α on M are 0 and ± 2. Therefore, if β is a root, then 2β is not a root. Hence $\left(\frac{1}{2}\right)\alpha$ is not a root. So, 1 is not a weight in M, i.e., M has no odd weights. Hence $M = \mathrm{Ker}(\alpha) + S_\alpha$. This proves the claim.

Remark: We have now completed the following classification for certain Lie algebras:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

with $\dim(L_\alpha) = 1$. This is a very rigid structure.

Example: ($L = \mathfrak{Sl}(3, F)$) Define

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \mid a, b \in F \right\}.$$

Define $E_{i,j}$ for $i < j$ to be the matrix with a 1 in the (i, j) position and 0s elsewhere. Define $F_{i,j} := E_{i,j}^{\mathrm{T}}$. In this example, we're only considering $E_{1,2}, E_{1,3}, E_{2,3}$ and the corresponding Fs. Define $H_{i,j} := [E_{i,j}, F_{i,j}]$. For example,

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: H_{1,2}.$$

Now,

$$L = H \oplus \bigoplus_{i<j} [E_{i,j} \oplus F_{i,j}].$$

For each i, j, $\langle E_{i,j}, F_{i,j}, H_{i,j} \rangle \cong \mathfrak{Sl}(2, F)$. So, this object has three (non-disjoint) copies of $\mathfrak{Sl}(2, F)$ in it.

Example: Next consider the adjoint action of S_α on L. We already know that $M \cong F^{\dim(H)-1} \oplus V(2)$ by S_α. Consider $\beta \in \Phi$ with $\beta \neq \pm\alpha$. Let $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$ be a sum of S_α-submodules. Since $(\beta + i\alpha)(h) = 1$ can happen for at most one i and $(\beta + i\alpha)(h) = 0$ can never happen, we see that K is simple. Hence $K \cong V(m)$ for some m. The weights are

$$\{(\beta + i\alpha)(h_\alpha) \mid i \in \mathbb{Z},\ L_{\beta+i\alpha} \neq 0\}.$$

The highest weight is $(\beta + q\alpha)(h_\alpha) = \beta(h_\alpha) + 2q = m$ where q is the largest integer such that $\beta + q\alpha \in \Phi$. The smallest integer is $(\beta - r\alpha)(h_\alpha) = \beta(h) - 2r = -m$, where r is the largest integer such that $\beta - r\alpha \in \Phi$. So,

$$K = \bigoplus_{i=-r}^{q} L_{\beta+i\alpha}.$$

Notation: The set

$$\{\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta, \dots, \beta + q_\alpha\}$$

is called the "α-string through β".

So,

$$L \cong \text{Ker}(\alpha) \oplus S_\alpha \oplus \bigoplus K_\beta$$

where $\bigoplus K_\beta$ is summed over distinct α-strings in Φ with $\beta \neq 0$.

1.15 Day 15- 10/03/12

Consider the decomposition

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha.$$

We have a Killing form $\kappa : H \times H \to F$ which is nondegenerate.

For $\delta \in H^*$, let $t_\delta \in H$ be defined by

$$\kappa(t_\delta, h) = \delta(h)$$

for all $h \in H$. So, for $\alpha \in \Phi$, we have $t_\alpha \in H$, and

$$\kappa(t_\alpha, h) = \alpha(h).$$

Let $\alpha in \Phi$ and let $x_\alpha \in L_\alpha$. Pick $y_\alpha \in L_{-\alpha}$ such that

$$[x_\alpha, y_\alpha] = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} =: h_\alpha.$$

Then, $x_\alpha, y_\alpha, h_\alpha$ satisfy the same relations as

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In particular, $[h, e] = 2e$, i.e., $[h_\alpha, x_\alpha] = 2x_\alpha$.

Recall that if $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$ then we can talk about the α-string through β. Consider

$$K = \bigoplus_{i\in\mathbb{Z}} L_{\beta+i\alpha},$$

a simple S_α module. Let q be the biggest such that $\beta + q\alpha \in \Phi$ and let r be the biggest such that $\beta - r\alpha \in \Phi$.

Since weights of simple S_α-modules form a finite arithmetic progression of step size 2, it follows that

$$\beta - r\alpha,\ \beta - (r-1)\alpha,\ \cdots,\ \beta + (q-1)\alpha,\ \beta + q\alpha.$$

The highest weight is
$$(\beta + q\alpha)(h_\alpha) = \beta(h_\alpha) + 2q.$$
The lowest weight is
$$(\beta - r\alpha)(h_\alpha) = \beta(h_\alpha) - 2r.$$
Hence,
$$\beta(h_\alpha) - 2r = -(\beta(h_\alpha) + 2q)$$
i.e.,
$$\beta(h_\alpha) = r - q \in \mathbb{Z}.$$
Also,
$$\beta - \beta(h_\alpha)\alpha \in \Phi,$$
for all $\alpha, \beta \in \Phi$. This turns out to be a very important fact.

Since κ is nondegenerate on H¡ we can use it to define a nondegenerate form on H^* by
$$(\gamma, \delta) := \kappa(t_\gamma, t_\delta).$$
Φ spans H^* and so there exists a basis $\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \Phi$ of H^*. Note that $\ell = \dim(H)$.

Let $\beta \in \Phi$, with $\beta = \sum_{i=1}^{\ell} c_i\alpha_i$ with $c_i \in F$. We claim that $c_i \in \mathbb{Q}$.

Proof: Let $\alpha \in \Phi$. Then,
$$(\beta, \alpha_j) = \sum_{i=1} \ell c_i(\alpha_i, \alpha_j).$$
Now multiply by $\dfrac{2}{(\alpha_j, \alpha_j)}$.
$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^{\ell} \frac{2(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)} c_i.$$
We claim that the left-hand side and each term on the right-hand side (excluding the c_i in each term) are integers. The result will then follow by looking at this as a system of equations in the variables c_i.

To see that these are integers,
$$\begin{aligned}\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} &= \frac{2\kappa(t_\beta, t_\alpha)}{(\alpha_j, \alpha_j)} \\ &= \kappa(t_\beta, h_{\alpha_j}) \\ &= \beta(h_{\alpha_j}) \in \mathbb{Z}.\end{aligned}$$
In the above, we use the fact that
$$h_{\alpha_j} = \frac{2t_{\alpha_j}}{(\alpha_j, \alpha_j)}.$$
Since the form $(\cdot, \cdot)$ is nondegenerate, the matrix (α_i, α_j) is invertible and hence so is the matrix $\dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$. Hence the system has a unique solution $(c_1, \ldots, c_\ell)$ and $c_i \in \mathbb{Q}$. $\square$

Define,
$$E_{\mathbb{Q}} := \text{ the } \mathbb{Q}\text{-span of } \Phi \subseteq H^*.$$
Observe that
$$\dim(E_{\mathbb{Q}}) = \ell.$$

We have the map

$$(\cdot,\cdot) : H^* \times H^* \to F.$$

We want a map

$$E_{\mathbb{Q}} \times E_{\mathbb{Q}} \to \mathbb{Q}.$$

Let $\lambda, \mu \in H^*$. Then,

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu).$$

To see the equality, consider $\kappa(x, y) = \operatorname{tr}_L(\operatorname{ad} x \operatorname{ad} y)$ on L decomposed as $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$. If $x, y \in H$ then the H term disappears, and over each L_α we have $\operatorname{tr}_{L_\alpha}(xy) = \alpha(x)\alpha(y)$.

In particular, if $\beta \in \Phi$, then

$$\begin{aligned}(\beta, \beta) &= \sum_{\alpha \in \Phi} \alpha(t_b)^2 \\ &= \sum_{\alpha \in \Phi} (\alpha, \beta)^2\end{aligned}$$

and so

$$\frac{1}{(\beta, \beta)} = \sum_{\alpha \in \Phi} \left(\underbrace{\frac{(\alpha, \beta)}{(\beta, \beta)}}_{\in \mathbb{Q}} \right)^2 .$$

So, $(\beta, \beta) \in \mathbb{Q}$ for all β,. Hence, $(\beta, \alpha) \in \mathbb{Q}$ for all α, β.

Now,

$$(\lambda, \lambda) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)^2 = \sum_{\alpha \in \Phi} (\alpha, \lambda)^2 > 0.$$

Let $E := \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$ so that we can extend $(\cdot,\cdot)\big|_{E_{\mathbb{Q}}}$ to E. E is a Euclidean space of dimension ℓ, and

$$(E, (\cdot,\cdot)) \supseteq \Phi.$$

Theorem:

(a) Φ spans E and $0 \notin \Phi$.

(b) If $\alpha \in \Phi$, then $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(c) If $\alpha, \beta \in \Phi$, then $\beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$.

(d) If $\alpha, \beta \in \Phi$, then $\dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

We've proved the components of this theorem already. In general, if E is a Euclidean space, these are the axioms for a root system in E.

1.16 Day 16 - 10/05/12

Remark: We now investigate arbitrary root systems, without considering whether or not they are derived from a Lie algebra. We will occasionally prove statements that are trivial if the root system is a Lie algebra, but not trivial for arbitrary root spaces.

Remark: Let $(E, (\cdots, \cdots))$ be a real Euclidean space of dimension ℓ. If $\alpha \in E$ is nonzero, then define σ_α to be reflection in the hyperplane $\alpha^\perp$. If $\beta \in E$, then

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)\alpha}{(\alpha, \alpha)}.$$

Notation: The textbook uses the notation

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

The problem with this notation is that $\langle \cdot, \cdot \rangle$ is not bilinear (it's not linear in the second variable), as the notation would suggest. So, instead, we define

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$$

so that

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)\alpha}{(\alpha, \alpha)} = \beta - (\beta, \alpha^\vee)\alpha.$$

(To see this equality, write out $\alpha^\vee$ and pull out the constants.)

Lemma: Let Φ be a finite set which spans E such that $\sigma_\alpha \Phi \subseteq \Phi$ for all $\alpha \in \Phi$. If $\sigma \in GL(E)$ leaves Φ invariant, fixes a hyperplane P pointwise, and sends some nonzero α to $-\alpha$, then $\sigma = \sigma_\alpha$, with $P = \alpha^\perp$.

Proof: Let $\tau = \sigma\sigma_\alpha$. Then, $\tau(\Phi) \subseteq \Phi$. Also, $\tau(\alpha) = \alpha$ because τ acts as the identity on E/R_α. Hence all eigenvalues of $\sigma\sigma_\alpha$ are 1. The matrix is of the form

$$\begin{array}{c|cccc} 1 & * & * & * & * \\ \hline & & & & \\ & & I & & \\ & & & & \end{array}$$

Since $\tau(\Phi) \subseteq \Phi$, there exists k such that $\tau^k(\beta) = \beta$, for all $\beta \in \Phi$. Since Φ spans E, we have $\tau^k = 1$. Hence the minimal polynomial of τ divides $t^k - 1$ and $(\tau - 1)^e$. Hence, the minimal polynomial of τ is $r - 1$, i.e., τ is the identity. $\square$

Definition: A subset $\Phi \subseteq E$ is a <u>root system</u> if:

(R1) Φ is finite, Φ spans E, and $0 \notin \Phi$.

(R2) For $\alpha \in \Phi$, $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(R3) For $\alpha \in \Phi$, $\sigma_\alpha(\Phi) = \Phi$.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$.

Definition: The subgroup $\langle \sigma_\alpha \mid \alpha \in \Phi \rangle \subseteq GL(E)$ is called the <u>Weyl group</u> of Φ.

Lemma: If $\sigma \in GL(E)$ leaves Φ invariant and $\alpha \in \Phi$, then

$$\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$$

and

$$(\beta, \alpha^\vee) = (\sigma(\beta), \sigma(\alpha)^\vee)$$

for all $\alpha, \beta \in \Phi$.

Proof: Well,

$$\sigma\sigma_\alpha\sigma^{-1}(\sigma(\alpha)) = \sigma\sigma_\alpha(\alpha) = -\sigma(\alpha).$$

Let P_α be the hyperplane perpendicular to α. Let $\sigma_x \in P_\alpha$ with $x \in P_\alpha$. Note that

$$\sigma(\sigma_\alpha)\sigma^{-1}(\sigma x) = \sigma x.$$

Thus $\sigma\sigma_\alpha\sigma^{-1}$ fixes $\sigma(P_\alpha)$ pointwise. By the earlier lemma,

$$\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}.$$

Now,

$$\begin{aligned}\sigma_{\sigma(\alpha)}(\sigma\beta) &= \sigma\sigma_\alpha\sigma^{-1}(\sigma\beta)\\ &= \sigma(\beta - (\beta, \alpha^\vee)\alpha)\\ &= \sigma(\beta) - (\beta, \alpha^\vee)\sigma(\alpha). \ \square\end{aligned}$$

Example: If $\ell = 1$, we just have the set $\{-\alpha, \alpha\}$ on a number line.

Example: If $\ell = 2$ then we have $(\beta, \alpha^\vee) \in \mathbb{Z}$. Observe that

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2|\beta|\cos(\theta)}{|\alpha|}.$$

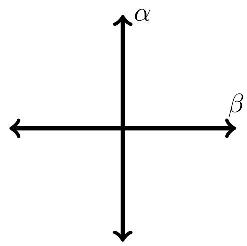
So,

$$(\beta, \alpha^\vee)(\alpha, \beta^\vee) = 4\cos^2(\theta) \leq 4$$

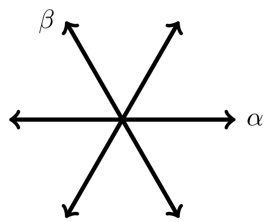
and this product must be an integer. So what are the possibilities? Assume without loss of generality that $|\beta| \geq |\alpha|$ and that $\alpha \neq \pm\beta$.

$(\alpha, \beta^\vee)$	$(\beta, \alpha^\vee)$	θ	$\frac{\lvert\beta\rvert^2}{\lvert\alpha\rvert^2}$
0	0	$\frac{\pi}{2}$	undefined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

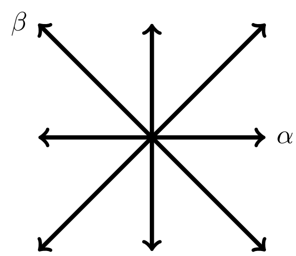
The first row corresponds to $A_1 \times A_1$ (and we have $W \cong Z_2 \times Z_2$):



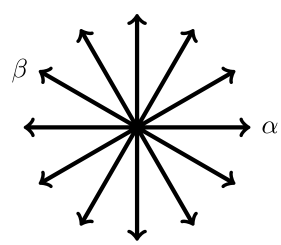
The second row corresponds to A_2 (and we have $W \cong S_3$):



The third row corresponds to B_2/C_2 (and we have $W \cong D_8$):



The sixth row corresponds to G_2 (and we have $W \cong D_{12}$):



In all of the above cases, $\sigma = \langle \sigma_\alpha, \sigma_\beta \rangle$.

1.17 Day 17 - 10/08/12

Lemma: Suppose $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$.

(i) If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.

(ii) If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.

Proof: Suppose $(\alpha, \beta) > 0$. Then, $(\alpha, \beta^\vee) > 0$ and $(\beta, \alpha^\vee) > 0$.

If $(\beta, \alpha^\vee) = 1$, then $\sigma_\alpha(\beta) = \beta - \alpha \in \Phi$. Thus, $\alpha - \beta \in \Phi$.

If $(\alpha, \beta^\vee) = 1$, then $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$. $\square$

Corollary: Let $\alpha, \beta \in \Phi$, with $\beta \neq \pm\alpha$. Let q be maximal such that $\beta + q\alpha$ is a root and let r be maximal such that $\beta - r\alpha$ is a root. Then, each element of $\{\beta + i\alpha \mid -r \leq i \leq q\}$ is also a root.

Proof: Assume there is a gap. Use the lemma above to show a contradiction. □

Remark: $\sigma_\alpha(x) = x - (x, \alpha^\vee)\alpha$, i.e., σ_α reverses the root string. We see that

$$\sigma_\alpha(\beta + q\alpha) = \beta - r\alpha.$$

So, $(\beta, \alpha^\vee) = r - q$. Suppose β was at the end of the α-string. Then, $q = 0$ and $r =$ the length of the string. So, $(\beta, \alpha^\vee)$ is the length of the string, which shows that the length of *any* root string is ≤ 4.

1.17.1 Bases

Definition: A subset $\Delta \subseteq \Phi$ is a base if

(B1) Δ is a base of E.

(B2) Each root β can be written as

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

where k_α integers which are either all nonnegative or all nonpositive.

Theorem: Φ has a base.

Definition: For $\alpha \in \Phi$ and $P_\alpha := \alpha^\perp$, say that $\gamma \in E$ is regular if $\gamma \in E \setminus \bigcup_\alpha P_\alpha$.

Proof: Fix a regular element γ. Define

$$\Phi^+(\gamma) := \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

and note that

$$\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma).$$

Let $\beta \in \Phi^+(\gamma)$. Say that β is decomposable if $\beta = \beta_1 + \beta_2$ where $\beta_i \in \Phi^+(\gamma)$ and say that β is indecomposable otherwise.

Claim: If $\gamma \in E$ is regular, then the set $\Delta(\gamma)$ of indecomposable elements of $\Phi^+(\gamma)$ is a base of Φ. Additionally, every base of Φ is of this form.

Proof:

(1) Each root in $\Phi^+(\gamma)$ is a nonnegative $\mathbb{Z}$-combination of $\Delta(\gamma)$.

Proof: Suppose not. Pick $\alpha \in \Phi^+(\gamma)$ which is not so written and such that (α, γ) is minimal. Then, $\alpha \notin \Delta(\gamma)$. So $\alpha = \beta_1 + \beta_2$ where $\beta_1, \beta_2 \in \Phi^+(\gamma)$. Then, $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. Hence, $(\gamma, \beta_i) < (\gamma < \alpha)$. Thus, $\beta_i \in \mathbb{Z}_+\Delta$ for $i = 1, 2$. Therefore, $\alpha \in \mathbb{Z}^+\Delta$, which is a contradiction.

(2) If $\alpha, \beta \in \Delta(\gamma)$ then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$.

Proof: If $\alpha \neq \beta$ and $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$. So, either $\alpha - \beta \in \Phi^+(\gamma)$ or $\beta - \alpha \in \Phi^+(\gamma)$. Rewrite $\alpha = (\alpha - \beta) + \beta$ and $\beta = (\beta - \alpha) + \alpha$. This contradicts $\alpha, \beta \in \Delta(\gamma)$.

(3) $\Delta(\gamma)$ is linearly independent.

Proof: Suppose $\sum_{\alpha\in\Delta(\gamma)} r_\alpha \alpha = 0$. Suppose we can write

$$\sum_{\alpha\in\Delta(\gamma)} s_\alpha \alpha = \sum_{\alpha\in\Delta(\gamma)} t_\beta \beta$$

with $s_\alpha, t_\beta > 0$. Then,

$$0 \leq (\epsilon, \epsilon) = \sum_{\alpha\in\Delta(\gamma)} s_\alpha t_\beta (\alpha, \beta) \leq 0.$$

Thus, $\epsilon = 0$ and so

$$0 = (\gamma, \epsilon) = \left(\gamma, \sum_{\alpha\in\Delta(\gamma)} s_\alpha \alpha\right) = \sum_{\alpha\in\Delta(\gamma)} s_\alpha (\gamma, \alpha).$$

Therefore, all $s_\alpha = 0$ and similarly all $t_\beta = 0$.

(4) $\Delta(\gamma)$ is a base.

Proof: See book.

(5) Each base Δ of Φ is of the form $\Delta(\gamma)$.

Proof: Choose $\gamma \in E$ such that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. γ is regular by (B2). Let Φ^+ be the positive roots with respect to Δ. Since γ is regular, $\Phi^+ \subseteq \Phi^+(\gamma)$ and similarly, $\Phi^- \subseteq \Phi^-(\gamma)$, and so equality must hold. Since $\Phi^+ = \Phi^+(\gamma)$, Δ clearly consists of indecomposable elements, i.e., $\Delta \subseteq \Delta(\gamma)$. But by dimensions, we must have $\Delta = \Delta(\gamma)$.

We have now proved the theorem. □

1.18 Day 18 - 10/10/12

Remark: Roots in Δ are called simple roots with respect to Δ. If $\alpha \in \Phi^+$ with

$$\alpha = \sum_{\beta\in\Delta} k_\beta \beta$$

then

$$\sum_{\beta\in\Delta} k_\beta$$

is called the height of α.

Definition: The connected components of $E \setminus \bigcup P_\alpha$ a re called Weyl chambers. The fundamental Weyl chamber with respect to Δ is

$$\mathcal{C}(\Delta) := \{x \in E \mid (x, \alpha) > 0 \text{ for all } \alpha \in \Delta\}.$$

(The condition would be equivalent if we said "for all $\alpha \in \Phi^+$".)

Remark: Recall that Δ is of the form $\Delta(\gamma)$ for some regular γ. By the decomposition of the fundamental Weyl chamber $\mathcal{C}(\Delta)$, we have $\gamma \in \mathcal{C}(\Delta)$. Also, for any $\gamma' \in \mathcal{C}(\Delta)$, we have $\Delta(\gamma') = \Delta(\gamma)$.

Lemma A: Fix a base Δ. If α is positive (i.e., all of the nonzero coefficients of α when expressed as a sum of simple root are positive) but not simple, then there exists $\beta \in \Delta$ such that $\alpha - \beta$ is a (positive) root.

> **Proof:** Suppose there exists $\beta \in \Delta$ such that $(\alpha, \beta) > 0$. Then, by an earlier lemma, $\alpha - \beta \in \Phi$. Since α not simple, it's a combination of at least two simple roots with positive coefficients. Since we're subtracting only one of them, the other one still has positive coefficient, and so all the coefficients must be positive. So, $\alpha - \beta$ is a positive root.
>
> Now suppose that no such β exists, i.e., $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$. We claim that the set $\Delta \cup \{a\}$ is linearly independent. (See book for proof of this claim.) This is a contradiction. So, there exists such a $\beta \in \Delta$. $\square$

Corollary: If $\alpha \in \Phi^+$, then α can be written as $\alpha = \alpha_1 + \cdots + \alpha_t$, for $\alpha_i \in \Delta$ and for all $i \in [t]$ with $\alpha_1 + \cdots + \alpha_i \in \Phi$.

Lemma B: Let $\alpha \in \Delta$. Then, σ_α permutes the set $\Phi^+ \setminus \{a\}$.

Notation: Let $\delta := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Corollary: $\sigma_\alpha(\delta) = \delta - \alpha$.

Notation: We use "$\alpha > 0$" to mean $\alpha \in \Phi^+$ and "$\alpha < 0$" to mean $\alpha \in \Phi^-$.

Lemma C: Let $\alpha_1, \ldots, \alpha_t \in \Delta$ (not necessarily distinct). Write $\sigma_i := \sigma_{\alpha_i}$. If

$$\sigma_1 \sigma_2 \cdots \sigma_{t-1}(\alpha_t) < 0,$$

then for some s with $1 \leq s < t$, we have

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}.$$

> **Proof:** Set $\beta_i := \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$ for $i \in [t-2]$, and define $\beta_{t-1} := \alpha_t$. Then, $\beta_0 < 0$ and $\beta_{t-1} > 0$. Then,
>
> $$\sigma_s(\beta_s) = \beta_{s-1} < 0.$$
>
> So, $\beta_s = \alpha_s$ be an earlier lemma. Now, if $\sigma(\gamma) = \delta$, then $\sigma \sigma_\gamma \sigma^{-1} = \sigma_\delta$. So,
>
> $$\underbrace{\sigma_{s+1} \cdots \sigma_{t-1}}_{\beta_s}(\alpha_t) = \alpha_s.$$
>
> We see that
>
> $$(\sigma_{s+1} \cdots \sigma_{t-1})(\sigma_t(\sigma_{t-1} \cdots \sigma_{s+1})) = \sigma_s$$
>
> and so
>
> $$\sigma_{s+1} \cdots \sigma_{t-1} = \sigma_s \sigma_{s+1} \cdots \sigma_t.$$
>
> This completes the proof. $\square$

Corollary: If $\sigma = \sigma_1 \cdots \sigma_t$ is the shortest possible expression for σ as a product of simple reflections, then $\sigma(\alpha_t) < 0$.

1.19 Day 19 - 10/12/12

Theorem: Let Δ be a base of Φ.

(a) If $\gamma \in E$ is regular, then there exists $\sigma \in W$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$, i.e. $\sigma(\gamma) \in \mathcal{C}(\Delta)$. So, W acts transitively on Weyl chambers.

(b) If Δ' is another base, then there exists $\sigma \in W$ such that $\sigma(\Delta') = \Delta$.

(c) If $\alpha \in \Phi$, then there exists $\sigma \in W$ such that $\sigma(\alpha) \in W$.

(d) $W = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$.

(e) If $\sigma(\Delta) = \Delta)$, then $\sigma = 1$.

Proof: Let $W' := \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. Prove parts (a), (b), and (c) with W' in place of W.

For part (a), let $\delta := \frac{1}{2} \sum_{\alpha > 0} \alpha$. Choose $\sigma \in W'$ so that $(\sigma(\gamma), \delta)$ is as big as possible. Let $\alpha \in \Delta$. Then,

$$\begin{aligned} (\sigma_a \sigma(\gamma), \delta) &= (\sigma(\gamma), \sigma_\alpha(\delta)) \\ &= (\sigma(\gamma), \delta) \\ &= (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha). \end{aligned}$$

Since γ is regular, $\sigma(\gamma)$ is also regular. So, $(\sigma(\gamma), \alpha) > 0$, i.e., $\sigma(\gamma) \in \mathcal{C}(\Delta)$.

Part (b) follows immediately.

In part (c), pick $\gamma_0 \in P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$. Then there exists $\epsilon > 0$ such that $(\gamma_0, \alpha) = 0$ and $(\gamma_0, \beta) > \epsilon$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$. Pick γ very close to γ_0 such that for some $\epsilon' > 0$,

$$0 < (\gamma, \alpha) < \epsilon'$$

while

$$|(\gamma, \beta)| > \epsilon'$$

for all $\beta \in \Phi^+ \setminus \{\alpha\}$. Then, $\alpha \in \Delta(\gamma')$. Since $\alpha \in \Phi^+(\gamma)$ and is indecomposable, $\alpha = \beta_1 + \beta_2$ with $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$ for $\beta_1, \beta_2 \in \Phi^+(\gamma)$.

In part (d), let $\alpha \in \Phi$ and pick $\sigma \in W'$ such that $\sigma(\alpha) = \beta \in \Delta$. Then, $\sigma_\beta = \sigma \sigma_\alpha \sigma^{-1}$. So, $\sigma_\alpha = \sigma^{-1} \sigma_\beta \sigma$, and each of the three maps on the right-hand side are in W'.

For part (e), suppose $\sigma(\Delta) = \Delta$. Write $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ as a product of *simple* reflections in the shortest possible way. Then, the earlier lemma shows that $\sigma(\alpha_t) < 0$, and therefore $\sigma = 1$. □

Definition: Let Δ be a base of Φ and $\sigma \in W$. We define the length of σ, denoted $\ell(\sigma)$ to be the shortest length of an expression for σ as a product of simple reflections. We call such a shortest expression reduced.

Lemma: For all $\sigma \in W$, $\ell(\sigma) =$ the number of positive roots α such that $\sigma(\alpha) < 0$.

Proof: Let $n(\sigma)$ denote the number of positive roots α such that $\sigma(\alpha) < 0$. We proceed by induction on $\ell(\sigma)$. If $\ell(\sigma) = 1$, then the proof is clear. Write σ in reduced form

$$\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}.$$

Then, $\sigma(\alpha_t) < 0$. Well, $n(\sigma \sigma_{\alpha_t}) = n(\sigma) - 1$ and $\ell(\sigma \sigma_{\alpha_t}) = \ell(\sigma) - 1$, so induction gives the result. □

Notation: Let $\overline{\mathcal{C}(\Delta)}$ denote the closure of $\mathcal{C}(\Delta)$ in E.

Lemma: Let $\lambda, \mu \in \overline{\mathcal{C}(\Delta)}$. If $\sigma(\lambda) = \mu$ for some $\sigma \in W$, then σ is a product of simple reflections, each of which fixes λ. In particular, $\lambda = \mu$.

> **Proof:** We proceed by induction on $\ell(\sigma)$. The base case is trivial. Suppose $\sigma \neq 1$. Then, σ must send some simple root to a negative root. Suppose $\alpha \in \Delta$ is such that $\sigma(\alpha) < 0$. Now,
>
> $$0 \geq (\mu, \sigma(\alpha)) = (\sigma^{-1}\mu, \alpha) = (\lambda, \alpha) \geq 0.$$
>
> Hence,
>
> $$(\lambda, \alpha) = 0,$$
>
> so σ_α fixes λ. Then,
>
> $$(\sigma\sigma_\alpha)(\lambda) = \sigma(\lambda) = \mu.$$
>
> Also, $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1$. Induction applies. $\square$

Corollary: $\overline{\mathcal{C}(\Delta)}$ is a fundamental domain for the action of W on E.

1.20 Day 20 - 10/15/12

Definition: A root space is irreducible if it can't be decomposed into nontrivial root spaces.

Recall: We say $x > y$ for roots x, y if either $x = y$ or $x - y \in \Phi^+$.

Lemma A: Let Φ be irreducible. Relative to $<$, there is a unique maximal root β. If

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

then $k_\alpha > 0$ for all α.

> **Proof:** Let β be a maximal root. It's clear that the maximal roots exist since the set is finite. What we must show is that β is unique. Well, we have that $\beta + \alpha \notin \Phi$ for all $\alpha \in \Delta$. So, we must have that $(\beta, \alpha) \geq 0$. Suppose that
>
> $$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha = \sum_{\alpha \in \Delta_1} k_\alpha \alpha + \sum_{\alpha \in \Delta_2} k_\alpha \alpha,$$
>
> where $\Delta_1 = \{\alpha \mid k_\alpha \neq 0\}$ and $\Delta_2 = \{\alpha \mid k_\alpha = 0\}$. To prove the last statement of the lemma, we need to prove that $\Delta_2 = \emptyset$.
>
> Let $\gamma \in \Delta_2$. Then, $(\gamma, \alpha) \leq 0$ for all $\alpha \in \Delta_1$. since $\gamma, \alpha \in \Delta$ and $\gamma \neq \alpha$. Then, $(\gamma, \beta) \leq 0$, and hence $(\gamma, \beta) = 0$. Therefore, $(\gamma, \alpha) = 0$ for all $\alpha \in \Delta_1$. So $(\Delta_1, \Delta_2) = 0$. Therefore, $\Delta_2 = \emptyset$.
>
> Now we show the first part of the lemma. Let β and β' be maximal roots. Then, $(\beta', \alpha) \geq 0$ for all $\alpha \in \Delta$. But, Δ spans E and so there exists $\alpha \in \Delta$ such that $(\beta', \alpha) > 0$. Therefore, $(\beta', \beta) > 0$ where $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$. Hence, $\beta' - \beta \in \Phi$ or else $\beta = \beta'$. Hence either $\beta' - \beta \in \Phi^+$ meaning $\beta < \beta'$ or $\beta - \beta' \in \Phi^+$ meaning $\beta > \beta'$. Either way is a contradiction. $\square$

Lemma B: If Φ is irreducible, then W acts irreducibly on E. (Hence any W-orbit on Φ spans E.)

Proof: Let E' be a nonzero W-invariant subspace of E. Write $E = E' \oplus E''$ where $E'' = E'^{\perp}$. Let $\alpha \in \Phi$ and $v \in E'$. Then,

$$E' \ni \sigma_\alpha(v) = v - (v, \alpha^\vee)\alpha.$$

So, either $\alpha \in E'$ or $\alpha \in E''$. $\square$

Lemma C: If Φ is irreducible, then there are at most two root lengths.

Proof: Let α and β be arbitrary roots. Then, there exists $\sigma \in W$ such that $(\sigma(\alpha), \beta) \neq 0$ by **Lemma B**. So, assume without loss of generality that $(\alpha, \beta) \neq 0$. Now,

$$\frac{|\beta|^2}{|\alpha|^2} \in \left\{1, 2, 3, \frac{1}{2}, \frac{1}{3}\right\}.$$

But if we had a third root γ we could get a root length of $\frac{3}{2}$ for example, which is not possible. $\square$

Fact: All roots of the same length are conjugate under W.

Lemma D: In an irreducible root system, the maximal root is long (if there are exactly two lengths).

Proof: Let $\alpha \in \Phi$. Let β be the unique maximal root. We show that $(\beta, \beta) \geq (\alpha, \alpha)$. Without loss of generality, $\alpha \in \overline{\mathcal{C}(\Delta)}$. Then, $\beta - \alpha > 0$. So, $(\gamma, \beta - \alpha) \geq 0$ for all $\gamma \in \overline{\mathcal{C}(\Delta)}$. Well, $\beta \in \overline{\mathcal{C}(\Delta)}$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$. If $\gamma = \beta$ then $(\beta, \beta - \alpha) \geq 0$ and so $(\beta, \beta) \geq (\beta, \alpha)$. If $\gamma = \alpha$ then $(\alpha, \beta - \alpha) \geq 0$ and so $(\alpha, \beta) \geq (\alpha)$. Thus,

$$(\beta, \beta) \geq (\beta, \alpha) = (\alpha, \beta) \geq (\alpha, \alpha). \ \square$$

Definition: Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. The <u>Cartan matrix</u> is the matrix (C_{ij}) where

$$C_{ij} = (\alpha_i, \alpha_j^\vee) = \left(\alpha_i, \frac{2\alpha_j}{(\alpha_j, \alpha_j)}\right).$$

1.21 Day 21 - 10/17/12

1.21.1 Cartan Matrices

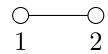
Consider $(\alpha_i, \alpha_j^\vee)$ with $\alpha_i, \alpha_j \in \Delta$. We have the following examples of Cartan matrices.

Proposition: If $\Phi' \subseteq E'$ is another root system with base $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$, then the bijection $\alpha_i \mapsto \alpha'_i$ extends to an isomorphism $E \to E'$ sending Φ to Φ'. Therefore, the Cartan matrix determines Φ up to isomorphism.

1.21.2 Coxeter Graphs and Dynkin Diagrams

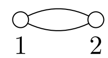
Definition: To draw the Coxeter graph, we draw vertices $\{1, \cdots, \ell\}$. We join i and j by $(\alpha_i, \alpha_j^\vee)$ and $(\alpha_j, \alpha_i^\vee)$ edges, where $i \neq j$.

Example: In the first example above, which is $\mathfrak{Sl}_3(F)$, the Coxeter graph is



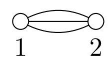
This information encodes the angle between α_1 and α_2 and also encodes the order of σ_{α_1} and σ_{α_2}. In this case, the angle between them is $2\pi/3$.

Example: In the second example above, the Coxeter graph is



The angle between them is $3\pi/4$.

Example: In the third example above, the Coxeter graph is



The angle between them is $5\pi/6$.

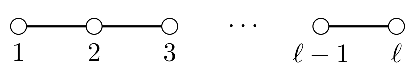
Example: In the fourth example, the Coxeter graph has no lines:

This means that the corresponding reflections commute. The angle between them is $\pi/2$.

Definition: In a Dynkin diagram, we take the Coxeter graph and draw an arrow which represents in inequality through the edges. In all the ones above, the arrow is $<$.

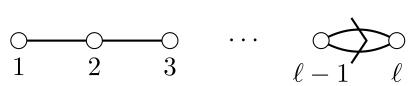
Classification Theorem: If Φ is an irreducible root system of rank ℓ, then its Dynkin diagram is one of the following:

- A_ℓ ($\ell \geq 1$):



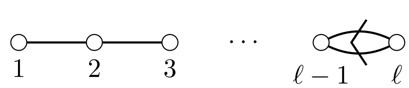
In this case, $W(A_\ell) \cong S_{\ell+1}$.

- B_ℓ ($\ell \geq 2$):



In this case $W(B_\ell) \cong Z_2^\ell \rtimes S_\ell$.

- C_ℓ ($\ell \geq 3$):

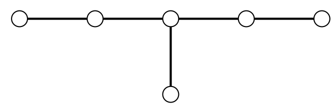


In this case $W(C_\ell) \cong Z_2^\ell \rtimes S_\ell$.

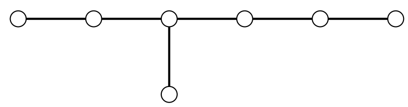
- D_ℓ ($\ell \geq 4$):

In this case $W(D_\ell) \cong Z_2^{\ell-1} \rtimes S_\ell$.

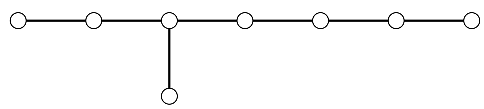
- E_6:



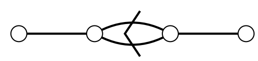
- E_7:



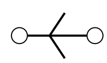
- E_8:



- F_4:



- G_2:



Proof: See book for proof. □

Description of A_ℓ: A_ℓ has orthogonal lines $e_1, \ldots, e_{\ell+1}$ in $\mathbb{R}^{\ell+1}$. We have

$$E = \langle e_1 + \cdots + e_{\ell+1} \rangle^\perp \cong \mathbb{R}^\ell,$$

$$\Phi = \{e_i - e_j \mid i \neq j\}, \quad |\Phi^+| = \binom{\ell+1}{2},$$

$$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_\ell - e_{\ell+1}\}.$$

Now note that

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} = \frac{2\alpha}{2} = \alpha.$$

The Cartan matrix has the form

$$\begin{pmatrix} 2 & -1 & - & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & -2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Description of B_ℓ: B_ℓ has orthogonal lines $e_1, \ldots, e_\ell$ in $\mathbb{R}^\ell$. We have

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\}_{i=1}^{\ell},$$

$$\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\}_{i=1}^{\ell},$$

$$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{\ell-1} - e_\ell, e_\ell\}.$$

Observe that

$$\left|\Phi^+\right| = 2\binom{\ell}{2} + \ell = \ell^2.$$

The Cartan matrix has the form

$$\left(\begin{array}{cccccc|c}
2 & -1 & - & \cdots & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & \vdots \\
0 & -1 & \ddots & \ddots & \cdots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & -1 & -2 & -1 & 0 \\
0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\
\hline
0 & \cdots & \cdots & \cdots & 0 & -1 & 2
\end{array}\right)$$

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Description of C_ℓ: C_ℓ and B_ℓ are very similar, except that

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}.$$

Description of D_ℓ: We have that

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j\},$$

and

$$\Delta = \{e_1 - e_2, \ldots, e_{\ell-1} - e_\ell, e_{\ell-1} + e_\ell\}.$$

Hence,

$$\Phi^+ = \{e_i \pm e_j \mid i \neq j\}.$$

As noted earlier,

$$W \cong Z_2^{\ell-1} \rtimes S_\ell.$$

1.22.1 Weights

Definition: Let $\Phi \subseteq E$ and denote the system of coroots by

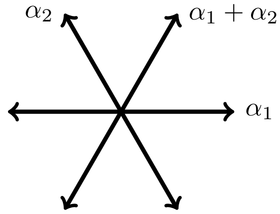
$$\Phi^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi.$$

Definition: Let $\Lambda_r = \mathbb{Z}\Phi$ be the root lattice and let $\Lambda_{cr} = \mathbb{Z}\Phi^\vee$ be the coroot lattice. Define

$$\Lambda := \{x \in E \mid (x, \alpha^\vee) \in \mathbb{Z}\ \forall \alpha \in \Phi\}.$$

This is the dual lattice of the coroot lattice. Λ is sometimes called the (abstract) weight lattice of Φ.

Remark: Since $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, we have that $\Lambda_r \subseteq \Lambda$. Consider A_2:



Here, we have $\Phi^\vee = \Phi$, $\alpha_1 = e_1 - e_2$, and $\alpha_2 = e_2 - 3_3$. Let λ_i be defined by

$$(\lambda_i, \alpha_j^\vee) = \delta_{ij}.$$

Then, the set $\{\lambda_i\}_i$ forms a basis for Λ. So, we know that

$$\alpha_1 = (1, -1, 0) \quad \text{and} \quad \alpha_2 = (0, 1, -1).$$

So, if $\lambda_1 = (a, b, -a-b)$, then we must have $a - b = 1$ and $b + a + b = 0$. This gives the solution $a = 2/3$ and $b = -1/3$. Thus,

$$\lambda_1 = (2/3, -1/3, -1/3).$$

By a similar calculation

$$\lambda_2 = (1/3, 1/3, -2/3).$$

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Definition: If $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$, define λ_i by $(\lambda_i, \alpha_j^\vee) = \delta_{i,j}$. Then, $\{\lambda_i\}_{i=1}^{\ell}$ forms a basis for Λ and the λ_i are called fundamental dominant weights. We also define

$$\Lambda_+ := \{\lambda \mid (\lambda, \alpha^\vee) \geq 0, \ \ \forall \alpha \in \Phi^+\}.$$

Remark: We have a partial ordering on Λ:

$$x \prec y \text{ if and only if } y - x \text{is a sum of positive roots.}$$

Warning: we can have $x \in \Lambda^+$ and $x \prec y$ but $y \notin \Lambda_+$.

Now, if we have a Lie algebra L and H is a maximal toral Lie subalgebra, say that

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Warning: this is not the same $\Phi \subseteq E$ as in our root system. This is $\Phi \subseteq H^*$. We had to carefully construct E from H^* earlier.

In each $L_\alpha, \alpha \in \Phi$, pick an arbitrary $x_\alpha \neq 0$. Then, there exists $y_\alpha \in L_{-\alpha}$ such that if $h_\alpha = [x_\alpha, y_\alpha]$, then the algebra $\langle x_\alpha, h_\alpha, y_\alpha \rangle \cong \mathfrak{Sl}(2, F)$ is such that

$$[h_\alpha, x_\alpha] = 2x_\alpha$$

and

$$[h_\alpha, y_\alpha] = -2y_\alpha.$$

H is spanned by h_α for $\alpha \in \Phi$ and $\{h_\alpha \mid \alpha \in \Delta\}$ is a basis. Note that

$$[h_\alpha, x_\beta] = (\beta, \alpha^\vee)x_\beta.$$

We see that $H \cong H^*$. For $t_\alpha \in H$, $(t_\alpha, h) = \alpha(h)$, with $h_\alpha = \dfrac{2t_\alpha}{(\alpha,\alpha)}$.

Consider

$$\Lambda_+ = \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0,\ \forall \alpha \in \Phi^+\}.$$

Each simple $\mathfrak{Sl}(2,F)$-module is of the form $V(n)$, where $V(n)$ has a basis $V_n, V_{n-2}, \cdots, V_{-n}$ and $hv_i = iv_i$. Thus, the simple $\mathfrak{Sl}(2,F)$-modules correspond to $n\lambda_1$ for $n \in \mathbb{N}_0$, where h corresponds to $\alpha_1^\vee$. So, $hv_n = nv_n$ means that v_n affords the eigenvalue $\langle n\lambda_1, \alpha^\vee\rangle$.

1.23.1 Representation Theory

Let L be a semisimple Lie algebra over F. Let H be a Cartan subalgebra (CSA) . Consider the roots Φ and the basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Write

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$$

with $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ and $h_\alpha = [x_\alpha, y_\alpha]$.

Definition: Let V be a finite dimensional L-modules. Then, H acts diagonally on V. Hence, we can write

$$V = \bigoplus_{\lambda\in H^*} V_\lambda,$$

where

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v\ \forall h \in H\}.$$

V_λ is called the weight space of V for the weight λ.

Remark: Warning: these are not the same weights as before from root spaces.

1.24 Day 24 - 10/24/12

Definition: A vector $v \in V_\lambda$ is called a maximal vector if $L_\alpha v = 0$ for all $\alpha > 0$.

Definition: The Borel subalgebra is

$$B(\Delta) = H + \sum_{\alpha>0} L_\alpha.$$

Recall: L-modules are equivalent to $\mathcal{U}(L)$-modules, where $\mathcal{U}(L)$ is the universal enveloping algebra.

Recall: Let L be a Lie algebra, and let T the tensor algebra of L (i.e., $T = \bigoplus_{i\geq 0} T_i$ where $T_i = \underbrace{L \otimes \cdots \otimes L}_{i \text{ times}}$).

If we consider

$$I = \langle x \otimes y - y \otimes x \mid x, y \in L\rangle \subseteq T,$$

then $T/I = S(L)$, where $S(L)$ is the symmetric algebra, which is isomorphic to the polynomial algebra on a basis of L. To get the universal enveloping algebra, we define

$$J = \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in L\rangle$$

and define $\mathcal{U} := T/J$.

1.24.1 Filtrations and Gradings

Define

$$T_m := T^0 \oplus T^1 \oplus \cdots \oplus T^m$$

and note that

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

and

$$T_i/T_{i-1} \cong T^i.$$

Let $U_{-1} = 0$ and $U_m = \pi(T_m)$, where π is the induced map in the commutative diagram below.

$$\begin{array}{ccc} L & \longrightarrow & T \\ \downarrow & \swarrow_{\pi} & \\ U = T/J & & \end{array}$$

Then, $U_m U_p \subseteq U_{m+p}$. Let $G^m := U_m/U_{m-1}$. This is a vector space. We have a map

$$G^m \times G^p \to G^{m+p}.$$

This makes

$$G := \bigoplus_{m=0}^{\infty} G^m$$

into a graded associative algebra.

Lemma: Consider $\phi_m : T^m \to U_m \to G^m$. This map is surjective (but $T^m \to U_m$ is not) and it induces an algebra homomorphism $\phi : T \to G$. Moreover, $\phi(I) = 0$, so we have an induced surjective homomorphism of algebras $\omega : S \to G$.

Theorem: (Poincaré-Birkhoff-Witt Theorem) $\omega : S \to G$ is an isomorphism of graded algebras.

Definition: V is a standard cyclic module if $V = \mathcal{U}(L)v^+$ for some maximal vector v^+. Well,

$$L = \underbrace{\sum_{\alpha<0} L_\alpha}_{n_i} \oplus \underbrace{H}_{h_i} \oplus \underbrace{\sum_{\alpha>0} L_\alpha}_{p_i}.$$

We have

$$p_i v^+ = 0,$$
$$h_i v^+ \in F_{v^+}.$$

1.25 Day 25 - 10/26/12

Consider a Lie algebra L with dimension n. Let

$$F = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$$

be a filtration of $\mathcal{U}(L)$. Let U_m be the image in U of

$$T_m = \underbrace{T^0}_{=F} \oplus \underbrace{T^1}_{=L} \oplus T^2 \oplus \cdots \oplus T^m \subseteq T(L).$$

By the **Poincaré-Birkhoff-Witt Theorem**, we have that $U_m/U_{m-1} \cong S^m(L)$. By using the "stars and bars" technique, we see that

$$\dim(S^m(L)) = \binom{m+n-1}{n-1} = \binom{m+n-1}{m}$$

(since we're counting the number of ways to pick a monomial of degree m from n different variables.)

Remark: Observe that

$$\text{gr}(U) = \bigoplus_m U_m/U_{m-1} \cong \bigoplus_m S^m(L) = S(L)$$

as graded algebras. Additionally, if $L = L_0 \oplus L_1$, then $S(L) \cong S(L_0) \otimes S(L_1)$.

Remark: If H is a subalgebra of L, then we claim that $\mathcal{U}(H) \hookrightarrow \mathcal{U}(L)$. Take a basis β of H and extend to a basis $\beta \cup \gamma$ of L. Then, $\mathcal{U}(H)$ has a basis of monomials in β and $\mathcal{U}(L)$ has a basis of elements of the form $m \cdot n$ where m is a monomial in β and n is a monomial in γ. This is by the **Poincar'e-Birkhoff-Witt Theorem**.

Remark: Suppose that L is semisimple. Then, we can write

$$L = \underbrace{\sum_{\alpha<0} L_\alpha}_{=:N} \oplus H \oplus \underbrace{\sum_{\alpha>0} L_\alpha}_{=:P}.$$

Now, $\mathcal{U}(L) \cong \mathcal{U}(N) \otimes \mathcal{U}(H) \otimes \mathcal{U}(P)$ as algebras. This is called the triangular decomposition.

Lemma: Let L be finite dimensional and semisimple. Let V be an L-module (possibly infinite dimensional). Let V_λ be the λ-weight space. Then,

(a) L_α maps V_λ into V_{λ_α} for $\alpha \in \Phi$.

(b) $V' = \sum_{\lambda \in H^*} V_\lambda$ is a direct sum and V' is an L-submodule of V.

(c) If $\dim(V) < \infty$, then $V = V'$.

Proof of (a): Suppose that $v \in V_\lambda$ and $x \in L_\alpha$. Then,

$$\begin{aligned} h(xv) &= (hx - xh + xh)v \\ &= ([h,x] + xh)v \\ &= (\alpha(h)x + xh)v \\ &= (\alpha(h) + \lambda(h))(xv) \\ &= (\alpha+\lambda)(h)(xv). \ \square \end{aligned}$$

1.25.1 Standard Cyclic Modules

Definition: Let L, Φ, Δ be as usual, and write

$$L = \bigoplus_{\alpha<0} L_\alpha \oplus H \oplus \bigoplus_{\alpha>0} L_\alpha.$$

Set $V = \mathcal{U}(L)v^+$ where v^+ is a maximal vector. Note that $L_\alpha v^+ = 0$ for $\alpha >)$ and $v^+ \in V_\lambda$ for some λ. We say that V is a standard cyclic module.

Theorem: Let V be a standard cyclic module with maximal vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$. Then,

(a) V is spanned by vectors $y\beta_1^{\ell_1}, \dots, y\beta_m^{\ell_m}$ where $y \in L_{-\alpha}$ (for $\alpha > 0$) and $\ell_i \in \mathbb{Z}^+$. In particular, V is the direct sum of its weight spaces.

(b) The weights of V are of the form

$$\mu = \lambda - \sum_{i=1}^{\ell} k_{\alpha_i} \alpha_i$$

for $k_i \in \mathbb{Z}^+$. So, all weights satisfy $\mu < \lambda$.

(c) For each $\mu \in H^*$, $\dim(V_\mu)$ is finite and $\dim(V_\lambda) = 1$. Also, $V_\lambda = \langle v^+ \rangle$.

(d) Each submodule of V is the direct sum of its weight spaces.

(e) V is an indecomposable L-module with a unique maximal submodule and irreducible quotient.

(f) Every nonzero homomorphic image of V is a standard cyclic module.

Corollary: Every finite dimensional irreducible module is standard cyclic.

1.26 Day 26 - 10/29/12

Theorem: Let W, V be standard cyclic modules of highest weight λ. If V and W are irreducible, then they are isomorphic.

> **Proof:** Let $X = V \oplus W$. Let v^+, w^+ be maximal vectors in V, W and define $x^+ := (v^+, w^+)$. Then, x^+ is a maximal vector in X. Let $Y := \mathcal{U}(L)x^+ \subseteq X$. Y is a standard cyclic module.
>
> Let $p : X \to V$ and $p' : X \to W$ be the projections. Then $p(x^+) = v^+$ and $p'(x+) = w^+$. So, $\mathrm{Im}(p)\big|_Y = V$ and $\mathrm{Im}(p')\big|_Y = W$. Hence, both V and W must be isomorphic to the unique simple quotient of Y, and hence $V \cong W$. $\square$

Remark: Our object now is to construct standard cyclic modules with highest weight λ^*. This is called the induced module construction.

Example: Let G be a group with $H \leq G$. Consider $FH \hookrightarrow FG$. For any FH-module M, the FG-module

$$FG \otimes_{FH} M$$

is the induced module.

Example: Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{f}$ a subalgebra. Then, for any left $\mathfrak{f}$-module (i.e., left $\mathcal{U}(\mathfrak{f})$-module), we can form the $\mathfrak{g}$-module

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{f})} M.$$

Construction: Consider rings R and S with 1 and a homomorphism $\phi : R \to S$. Then, ${}_S S_R$ is an (S, R) bimodule by the action

$$(s, r)(a) = (sa)\phi(r) = s(a\phi(r)).$$

Now, write the Lie algebra L as

$$L = \sum_{\alpha<0} L_\alpha \oplus H \oplus \sum_{\alpha>0} L_\alpha.$$

Set

$$B := B(\Delta) = H \oplus \underbrace{\sum_{\alpha>0} L_\alpha}_{=: P}$$

be a submodule. Note that P is an ideal of B and that $B/P \cong H$. Let $\lambda \in H^*$ and consider

$$B \longrightarrow H \xrightarrow{\lambda} F$$

We can consider λ as a homomorphism from B to F. Let $F_\lambda = \langle v^+ \rangle$ denote the 1-dimensional module on which B acts by

$$b \cdot x = \lambda(b) \cdot x$$

for $x \in F_\lambda$. Next define

$$Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} F_\lambda.$$

We claim that $Z(\lambda)$ is a standard cyclic module with highest weight λ, maximal vector $1 \otimes v^+$, such that $\mathcal{U}(L)$ is free as a right $\mathcal{U}(B)$-module. Now,

$$\mathcal{U}(L) \cong \mathcal{U}(B) \otimes_F S(L/B)$$

as $\mathcal{U}(B)$-modules. By construction, $Z(\lambda)$ is generated by $1 \otimes v^+$.

If $\alpha > 0$ and $x \in L_\alpha \subseteq B$, then

$$x(1 \otimes v^+) = x \otimes v^+ = 1 \otimes xv^+ = 0.$$

So, $1 \otimes v^+$ is killed by all L_α, $\alpha > 0$. For $h \in H$,

$$h(1 \otimes v^+) = h \otimes v^+ = 1 \otimes hv^+ = 1 \otimes \lambda(h)v^+ = \lambda(h)(1 \otimes v^+).$$

So, $1 \otimes v^+$ is indeed a maximal vector with highest weight λ.

Now, write

$$Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} F_\lambda.$$

By PBW,

$$\mathcal{U}(L) = \mathcal{U}(N) \otimes_F \mathcal{U}(B).$$

Hence,

$$\begin{aligned} Z(\lambda) &= \mathcal{U}(N) \otimes_F \mathcal{U}(B) \otimes_{\mathcal{U}(B)} F_\lambda \\ &= \mathcal{U}(N) \otimes_F F_\lambda \end{aligned}$$

as $\mathcal{U}(N)$-modules. So, $Z(\lambda)$ has a basis of the form

$$y_{\beta_1}^{i_1}, \ldots, y_{\beta_m}^{i_m} \otimes v^+$$

where $\beta_1, \ldots, \beta_m$ are the positive roots and $i_1, \ldots, i_m$ are nonnegative integers.

Definition: Let $V(\lambda)$ be the unique irreducible quotient of $Z(\lambda)$. Then, we have constructed a unique irreducible standard cyclic module $V(\lambda)$ for each $\lambda \in H^*$.

Question: Every finite irreducible module must be of the form $V(\lambda)$ for some $\lambda \in H^*$ (since it contains a maximal vector). For which $\lambda \in H^*$ is $V(\lambda)$ finite dimensional?

Remark: We can half-answer this question with the following necessary condition. If $V(\lambda)$ is finite dimensional, then $V(\lambda)$ is a finite dimensional module for each

$$S_\alpha \cong \mathfrak{Sl}_2(F)$$

for $\alpha \in \Phi^+$. Hence, from what we know about representations of $\mathfrak{Sl}_2(F)$, we must have

$$\lambda(h_\alpha) \in \mathbb{Z}^+$$

for $\alpha \in \Phi^+$.

Definition: $\lambda \in H^*$ is a dominant integral weight if $\lambda(h_\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Phi^+$.

Definition: Define $H_\mathbb{Z} := \langle h_i \mid i \in \Delta \rangle_\mathbb{Z} \subseteq H$. These are the dominant integral weights which are elements of the dual lattice to $H_\mathbb{Z}$ in H^*.

1.27 Day 27 - 10/31/12

Recall: $V(\lambda)$ is the irreducible standard cyclic module with highest weight $\lambda \in H^*$.

Definition: We say that λ is integral if $\lambda(h_{\alpha_i}) \in \mathbb{Z}$ for $\{\alpha_1, \ldots, \alpha_\ell\} = \Delta$.

Remark: The set of integral weights forms a $\mathbb{Z}$-lattice in H^* dual to the lattice $\mathbb{Z}\Delta \subseteq H$. We can identity the integral weights with the lattice $\Lambda \subseteq E$, where E is a Euclidean space.

Definition: Define
$$\Lambda^+ = \{\lambda \in \Lambda \mid \lambda(h_i) \geq 0\}$$
for all $i = 1, \ldots, \ell$. These are called the dominant weights. Note that we also have
$$\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0,\ \forall \alpha \in \Phi^+\}.$$

Theorem: If $\lambda \in H^*$ is dominant integral, then the irreducible L-module $V = V(\lambda)$ is finite dimensional and its sets of weights $\Pi(\lambda)$ is permuted by W with $\dim(V_\mu) = \dim(V_{\sigma\mu})$ for all $\sigma \in W$.

Corollary: The map $\lambda \mapsto V(\lambda)$ is a one-to-one correspondence from Λ^+ to the set of isomorphism classes of finite dimensional irreducible L-modules.

Lemma: (Properties of The Universal Enveloping Algebra) Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let $x_i, y_i h_i$ be standard generators of $S_i \cong \mathfrak{Sl}_2(F) \subseteq L$ (for $i \in [\ell]$). Then,

(a) $[x_j, y_i^{k+1}] = 0$ for all $i \neq j$.

Proof:
$$\begin{aligned}[x_j, y_i^{k+1}] &= x_j y_i^{k+1} - y_i^{k+1} x_j \\ &= (x_j y_i) y_i^k - (y_i x_j) y^k + (y_i x_j) y_i^k - y_i y_i^k x_j \\ &= [x_j, y_i] y_i^k + y_i [x_j, y_i^k].\end{aligned}$$

The first term is zero because it is contained in $L_{\alpha_i - \alpha_j}$, which is zero. The second term is zero by induction. □

(b) $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j) y_i^{k+1}$

(c) $[x_i, y_i^{k+1}] = -(k+1) y_i^k (k \cdot 1 - h_i)$

Theorem: Let $\phi : L \to \mathfrak{gl}(V)$ be the representation of L on $V = V(\lambda)$. Let $v^+ \in V_\lambda$ be a maximal vector. Let $m_i := \lambda(h_i)$. The $m_i \in Z^+$ by assumption.

(1) $y_i^{m_i+1} v^+ = 0$.

Proof: Let $w = y_i^{m_i+1}v^+$. By part (a) of the previous lemma, $x_j w = 0$ for $j \neq i$. So,

$$\begin{aligned} x_i w &= x_i y_i^{m_i+1} v^+ \\ &= y_i^{m_i+1} x_i v^+ - (m_i+1) y_i^{m_i}(m_i \cdot 1 - h_i)v^+. \qquad = y_i^{m_i+1} x_i v^+ \qquad (m_i \cdot 1 - h_i)v^+ = 0) \end{aligned}$$

So, if $w \neq 0$ then it would be a maximal vector, but off the wrong weight, this is a contradiction. □

(2) $\text{Span}(v^+, y_i v^+, \cdots, y_i^{m_i} v^+)$ is a nonzero S_i0submodule (finite dimensional) in V.

(3) V is the sum of finite dimensional S_i modules.

Proof: Let $V' \subseteq V$ be the sum of all S_i-submodules of V. Then, $V' \neq 0$ by (2). Let W be any finite dimensional S_i-submodule. The spam of $\{x_\alpha W, h_i W\}$ for $\alpha \in \Phi$ and $\alpha_i \in \Delta$ is a finite dimensional and an S_i-submodule. So, V' is stable. Hence, $V' = V$ since V is irreducible. □

(4) For $1 \leq i \leq \ell$, $\phi(x_i)$ and $\phi(y_i)$ are locally nilpotent and in $\text{End}(V)$. Every element of V lies in some finite dimensional S_i-module W and $\phi(x_i)\big|_W$ and $\phi(y_i)\big|_W$ are nilpotent.

(5) Let $S_i L = \exp(\phi(x_i))\exp(\phi(-y_i))\exp(\phi(x_i))$. S_i is a well-defined automorphism of V.

(6) If μ is any weight of V, then $S_i(V_\mu) = V_{\sigma_i \mu}$ where σ_i is the reflection with respect to α_i.

(7) $\Pi(\lambda)$ is stable under W.

(8) $\Pi(\lambda)$ is finite. Indeed, the set of W-conjugates of all dominant integral functions $\mu < \lambda$ is finite.

(9) $\dim(V) < \infty$¡ and so $\Pi(\lambda)$ is finite and for each $\mu \in \Pi(\lambda)$, $\dim(V_\mu) < \infty$. Since

$$V = \bigoplus_{\mu \in \Pi(\lambda)} V_\mu$$

uj we conclude $\dim(V) < \infty$.

1.28 Day 28 - 11/02/12

Example: Let $L := \mathfrak{Sl}(n, F)$ be the set of $n \times n$ matrices of trace 0. Let H be the diagonal subalgebra. Let $V = F^n$ be the natural module. We have the following (any matrix elements not shown are 0):

$$h_1 = \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix},$$

$$x_{\alpha_1} = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \quad x_{\alpha_2} = \begin{pmatrix} 0 & & & & 0 \\ & 0 & 1 & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix},$$

$$[h_1, x_\alpha] = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} = 2x,$$

etc. Now note that V is irreducible. Suppose $\{v_1, v_2, \cdots, v_n\}$ is the standard basis,

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \cdots \quad v_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Well,

$$\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, $h_1 v_1 = v_1$. Hence, v_1 is a weight vector for H, corresponding to weight ω_1. If $i > 1$, we can see that $h_i v_1 = 0$. Next, $h_1 v_2 = -v_2$ and $h_2 v_2 = v_2$. So, v_2 is a weight vector with weight $\omega_2 - \omega_1$. Continuing the calculation, we see that v_i for $i < n$ is a weight vector with weight $\omega_i - \omega_{i-1}$. Lastly, v_n is a weight vector with weight $-\omega_{n-1}$.

To check that v_1 is a maximal vector, we verify that

$$\begin{pmatrix} 0 & & & & \\ & 0 & & * & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, v_1 is a maximal vector, and $V = V(\omega_1)$.

To construct $V(\omega_i)$, consider the exterior powers $\Lambda^i V$.

Definition: If V is any vector space, then the <u>exterior algebra</u> is

$$T(V)/\langle v \otimes v \mid v \in V \rangle$$

where $T(V)$ is the <u>tensor algebra</u> defined by

$$T(V) = \bigoplus_{i \geq 0} T^i(V)$$

where

$$T_i(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{i \text{ terms}}.$$

$T(V)$ is a graded ring and $\langle v \otimes v \mid v \in V \rangle$ is homogeneous. Thus, the exterior algebra is graded and can be written

$$\Lambda(V) = \bigoplus_{i \geq 0} \Lambda^i(V).$$

1.29 Day 29 - 11/05/12

Recall: Define

$$\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0, \ \forall \alpha \in \Delta\}.$$

Then, $V(\lambda)$ is a simple finite dimensional module with highest weight $\lambda \in \Lambda^+$.

Notation: $\Pi(\lambda) := \{\mu \mid V(\lambda)_\mu \neq 0\}$.

Objective: For $\mu \in \Pi(\lambda)$, we want to find $\dim(V(\lambda)_\mu)$, called the weight multiplicity of μ in $V(\lambda)$.

Definition: A set of weights $\Pi \subseteq \Lambda$ is saturated if for all $\lambda \in \Pi$, $\alpha \in \Phi$, and $0 \leq i \leq (\lambda, \alpha^\vee)$, we have that the weight $\lambda - i\alpha \in \Pi$. We say that a saturated set Π has highest weight $\lambda \in \Lambda^+$ if $\mu < \lambda$ for all $\mu \in \Phi$.

Note: If Π is saturated, then Π is invariant under W, since

$$\sigma_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$

Lemma: Let Π be a saturated set of weights with highest weight $\lambda \in \Lambda^+$. Then, Π is finite.

Lemma: For $\lambda \in \Lambda^+$, $\Pi(\lambda)$ – the set of weights of $V(\lambda)$ – is saturated with highest weight λ.

Lemma: Let Π be a saturated set of weights with highest weight $\lambda \in \Lambda^+$. If $\mu \in \Lambda^+$ and $\mu < \lambda$, then $\mu \in \Pi$.

Proof: Let $\mu' := \mu + \sum_{\alpha \in \Delta} k_\alpha \alpha$, where $k_\alpha \geq 0$. Suppose that $\mu' \in \Pi$. We will show that for some $\alpha \in \Delta$ with $k_\alpha > 0$, we can lower k_α by 1 and the resulting weight is still in Π. Then, starting with $\lambda \in \Pi$ and repeating, we get $\mu \in \Pi$. We can assume that $\mu \neq \mu'$. So, there exists α with $k_\alpha > 0$. Hence,

$$0 < \left(\sum_{\alpha \in \Delta} k_\alpha \alpha, \ \sum_{\alpha \in \Delta} k_\alpha \alpha \right).$$

So, there exists $\beta \in \Delta$ such that $k_\beta > 0$ and

$$\left(\sum_{\alpha \in \Delta} k_\alpha \alpha, \beta \right) > 0$$

and hence

$$\left(\sum_{\alpha \in \Delta} k_\alpha \alpha, \beta^\vee \right) \geq 0.$$

Also, $(\mu, \beta^\vee) \geq 0$. Since $\beta \in \Delta$, $\mu \in \Lambda^+$. Hence,

$$(\mu', \beta^\vee) = \left(\mu + \sum_{\alpha \in \Delta} k_\alpha \alpha, \beta^\vee \right) > 0$$

since $\mu' \in \Pi$ and Π is saturated. So, $\mu' - \beta \in \Pi$. $\square$

Corollary: $\Pi(\lambda) = \{\sigma\mu \mid \mu \in \Lambda^+, \ \mu < \lambda, \ \sigma \in W\}$.

Remark: We can derive important formulas like **Freudenthal's Multiplicity Formula**, **Kostant's Formula** and **Weyl's Character Formula**.

Definition: One important weight that appears in all of the formulae is

$$\delta := \frac{1}{2}\sum_{\alpha\in\Phi}\alpha.$$

This has the property that

$$\sigma_\alpha(\delta) = \delta - \alpha.$$

Lemma: Let λ_i for $i \in [\ell]$ be the i^{th} fundamental dominant weight (i.e., $(\lambda_i, \alpha_j^\vee) = \delta_{i,j}$). Then,

$$\delta = \sum_{i=1}^{\ell}\lambda_i.$$

In particular, $(\delta, \alpha^\vee) > 0$ for all $\alpha \in \Phi^+$.

Proof: First note that

$$\begin{aligned}(\delta - \alpha_i, \alpha_i) &= (\sigma_i(\delta), \alpha_i)\\ &= (\sigma_i^2(\delta), \sigma_i(\alpha_i))\\ &= (\delta, -\alpha_i)\\ &= -(\delta, \alpha_i).\end{aligned}$$

So, $2(\delta, \alpha_i) = (\alpha_i, \alpha_i)$. And hence

$$\left(\delta, \frac{2\alpha_i}{(\alpha_i, \alpha_i)}\right) = 1,$$

i.e.,

$$(\delta, \alpha^\vee) = 1$$

for all i. Therefore,

$$\delta = \lambda_1 + \cdots + \lambda_\ell. \ \square$$

Lemma: Let $\mu \in \Lambda^+$ and $\nu = \sigma^{-1}\mu$ for some $\sigma \in W$. Then,

$$(\nu + \delta, \nu + \delta) \leq (\mu + \delta, \mu + \delta)$$

with equality only if $\mu = \nu$.

Proof: Well,

$$\begin{aligned}(\nu + \delta, \nu + \delta) &= (\sigma(\nu + \delta), \sigma(\nu + \delta))\\ &= (\mu + \sigma\delta, \mu + \sigma\delta)\\ &= (\mu + \delta, \mu + \delta) - 2(\mu, \delta - \sigma\delta).\end{aligned}$$

Since $\sigma\delta < \delta$ (look at modules with highest weight δ), we have that $\delta - \sigma\delta$ is the sum of positive roots. So, the right-hand term is nonnegative. This gives the inequality

$$(\nu + \delta, \nu + \delta) \leq (\mu + \delta, \mu + \delta).$$

We have equality if and only if $(\mu, \delta - \sigma\delta) = 0$, which occurs if and only if $(\mu, \delta) = (\mu, \sigma\delta) = (\nu, \delta)$. To see this, note that $(\mu - \nu, \delta) = 0$ and $\mu - \nu$ is a sum of positive roots. So, equality holds if and only if $\mu = \nu$. $\square$

1.30 Day 30 - 11/07/12

Definition: Let Π be a set of vectors. We say that Π is saturated if for all $\mu \in \Pi$, $0 \leq i \leq (\mu, \alpha^\vee)$, and $\alpha \in \Phi$, we have $\mu - i\alpha \in \Pi$.

Definition: A saturated set Π has highest weight λ if $\mu < \lambda$ for all $\mu \in \Pi$. If $\lambda \in \Lambda^+$, then

$$\Pi(\lambda) = \{\mu \in \Lambda \mid V(\lambda)\mu \neq 0\}$$

is a saturated set of weights with highest weight λ.

Lemma: If Π is saturated with highest weight λ and $\mu \in \Lambda^+$ and $\mu < \lambda$, then $\mu \in \Pi$.

Corollary: $\Pi(\lambda) = \{\sigma\mu \mid \mu \in \Lambda^+, \mu < \lambda, \sigma \in W\}$.

Definition: Define

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \lambda_1 + \cdots + \lambda_\ell$$

where $\lambda_i \in \Lambda^+$ is defined by $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. Recall that these are called the fundamental dominant weights.

Lemma: Let $\mu \in \Lambda^+$ and $\nu := \sigma^{-1}\mu$ for some $\sigma \in W$. Then,

$$(\mu + \delta, \mu + \delta) \leq (\nu + \delta, \nu + \delta)$$

with equality if and only if $\mu = \nu$.

Lemma: Let Π be a saturated set of weights with highest weight $\lambda \in \Lambda^+$. If $\mu \in \Pi$, then

$$(\mu + \delta, \mu + \delta) \leq (\lambda + \delta, \lambda + \delta).$$

Proof: By the previous lemma, we can assume that $\mu \in \Lambda^+$. Let $\lambda = \mu + \Sigma$ where Σ is the sum of the positive roots. Then,

$$\begin{aligned}(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta) &= (\lambda + \delta, \lambda + \delta) - (\lambda + \delta - \Sigma, \lambda + \delta - \Sigma) \\ &= (\lambda + \delta, \Sigma) + (\Sigma, \mu + \delta) \\ &\geq (\lambda + \delta, \Sigma) \\ &\geq 0\end{aligned}$$

with equality if and only if $\Sigma \neq 0$. $\square$

1.30.1 Freudenthal's Multiplicity Formula

Theorem: Let $m(\mu) = \dim(V(\lambda)_\mu)$ for $\mu \in \Lambda$. The $m(\mu)$ are given recursively by

$$((\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta))\, m(\mu) = 2 \sum_{\alpha > 0} \sum_{i=1}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha).$$

The starting point of the recursion is $m(\lambda) = 1$.

Example: Let $L = \mathfrak{Sl}(3, F)$. Then,

$$\Phi^+ = \{\alpha_1 = (1, -1, 0),\ \alpha_2 = (0, 1, -1),\ \alpha_1 + \alpha_2 = (1, 0, -1)\}.$$

We know $\Lambda \subseteq (1, 1, 1)^\perp$, and we calculate that

$$\begin{aligned}
\lambda_1 &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) & \lambda_2 &= \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \\
\lambda_1 &= \frac{2}{3}\alpha_1 + \alpha_2 & \lambda_2 &= \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 \\
\alpha_1 &= 2\lambda_1 - \lambda_2 & \alpha_2 &= 2\lambda_2 - \lambda_1 \\
\delta &= (1, 0, -1).
\end{aligned}$$

Choose $\lambda = \lambda_1 + 3\lambda_2 = \left(\frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\right)$. We calculate the first level of recursion

$$\begin{aligned}
\mu = \lambda - \alpha_1 &= (\lambda_1 + 3\lambda_2) - (2\lambda_1 - \lambda_2) = -\lambda_1 + 4\lambda_2 = \left(\frac{2}{3}, \frac{5}{3}, -\frac{7}{3}\right) \\
&= \lambda - \alpha_2 = (\lambda_1 + 3\lambda_2) - (2\lambda_2 - \lambda_1) = 2\lambda_1 + \lambda_2 = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right).
\end{aligned}$$

We have

$$(\lambda_1, \alpha_1^\vee) = 1, \qquad \text{and} \qquad (\lambda, \alpha_2^\vee) = 3.$$

Hence,

$$\lambda + \delta = \left(\frac{8}{3}, \frac{2}{3}, -\frac{10}{3}\right) \qquad \mu + \delta = \left(\frac{5}{3}, \frac{5}{3}, -\frac{10}{3}\right).$$

Thus,

$$((\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta)) = \frac{168}{9} - \frac{150}{9} = \frac{18}{9} = 2.$$

Then,

$$\begin{aligned}
2m(\mu) &= 2\sum_{\alpha > 0}\sum_{i=1}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha) \\
&= 2m(\lambda)(\lambda, \alpha_1) \\
&= 2m(\lambda) \cdot 1.
\end{aligned}$$

Therefore, $m(\mu) = m(\lambda) = 1$.

1.31 Day 31 - 11/14/12

Theorem: (Freudenthal's Formula) Consider $V(\lambda)$ with $\lambda \in \Lambda^+$. Define $m(\mu) := \dim(V(\lambda)_\mu)$ and

$$\delta := \frac{1}{2}\sum_{\alpha > 0} \alpha = \lambda_1 + \cdots + \lambda_\ell,$$

where $(\lambda_i, \alpha_j^\vee) = \delta_{i,j}$ and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Then

$$m(\mu)\left[(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta)\right] = 2\sum_{\alpha > 0}\sum_{i=1}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha).$$

Additionally, for $\mu \in \Pi(\lambda)$, we have that

$$\{\nu \mid \mu < \nu < \lambda\} \text{ is finite.}$$

Proof: (Outline)

(1) Consider a Casimir element in the center of $\mathcal{U}(L)$. It is of the form

$$c_L = \sum_i x_i y_i$$

where $\{x_i\}_i$ is a basis of L and $\{y_i\}_i$ is a dual basis with respect to some nondegenerate associative form (in this case, we'll use the Killing form). This element will play a similar role as in the proof of Weyl's Theorem. c_L acts on any $V(\lambda)$ as a scalar, say c. If

$$L = H \oplus \bigoplus_\alpha L_\alpha,$$

then the specific form of c_L is

$$\sum_{i=1}^{\ell} h_i k_i + \sum_{\alpha \in \Phi} x_\alpha z_\alpha$$

where $\{h_i\}_i$ and $\{x_\alpha\}_\alpha$ are standard bases and $\{k_i\}_i$ and $\{z_\alpha\}_\alpha$ are dual bases with respect to the Killing form.

Note that the terms $h_i k_i$ and $x_\alpha z_\alpha$ each send every weight space V_μ to itself. So, c_L preserves each V_μ.

(2) Compute the traces of each $h_i k_i$ and $x_\alpha z_\alpha$ on each V_μ. It's easy to calculate the traces of the $h_i k_i$. The other calculations are difficult.

To compute the trace of $x_\alpha z_\alpha$ on V_μ, we consider $V(\lambda)$ as a module for $S_\alpha = \langle x_\alpha, z_\alpha, h_\alpha \rangle = \mathfrak{Sl}(2, F)$. Let κ be the Killing form on L. Let $\{h_1, \ldots, h_\ell\}$ be a basis for K and let $\{k_1, \ldots, k_\ell\}$ be a dual basis with respect to $\kappa\big|_{H \times H}$. Pick nonzero $x_\alpha \in L_\alpha$ to be arbitrary. Then, there exists a unique $z_\alpha \in L_{-\alpha}$ satisfying

$$\kappa(x_\alpha, z_\alpha) = 1.$$

Before, we had standard generators x_α, y_α, and h_α of S_α. If we choose the same x_α, then we get that

$$\begin{aligned} z_\alpha &= \frac{(\alpha,\alpha)}{2} y_\alpha, \\ t_\alpha &= [x_\alpha, z_\alpha] = \frac{(\alpha,\alpha)}{2} h_\alpha, \\ [x_\alpha, z_\alpha] &= \frac{(\alpha,\alpha)}{2} h_\alpha, \\ [x_\alpha, y_\alpha] &= h_\alpha. \end{aligned}$$

So,

$$c_L = \sum_{i=1}^{\ell} h_i k_i + \sum_{\alpha \in \Phi} x_\alpha z_\alpha \in \mathcal{U}(L).$$

It must be checked that $c_L \in Z(\mathcal{U}(L))$.

Consider an irreducible S_α-module of highest weight $m \in \mathbb{N}$, with $h_\alpha v_0 = m v_0$. Let $\{v_0, \ldots, v_m\}$ be the basis we used for the module earlier, where v_0 was the vector killed by x_α and $h v_0 = m v_0$, and $v_i = \dfrac{y_\alpha^i}{i!} v_0$. Now, we have

$$\begin{aligned}
h_\alpha v_i &= (m-2i)v_i,\\
y_\alpha v_i &= (i+1)v_{i+1},\\
x_\alpha v_i &= (m-i+1)v_{i-1},\\
t_\alpha v_i &= (m-2i)\left(\frac{(\alpha,\alpha)}{2}\right)v_i,\\
z_\alpha v_i &= v_{i+1},\\
x_\alpha v_i &= i(m-i+1)\left(\frac{(\alpha,\alpha)}{2}\right)v_{u-1}.
\end{aligned}$$

So,

$$(x_\alpha z_\alpha)v_i = (m-i)(i+1)\left(\frac{(\alpha,\alpha)}{2}\right)v_i.$$

Let $\mu \in \Pi(\lambda)$ be such that $\mu + \alpha \notin \Pi(\lambda)$ for all $\alpha > 0$. For each such μ, we get a simple S_α submodule of $V(\lambda)$ of highest weight $m = (\mu, \alpha^\vee)$.

The α-string of weights through μ is

$$\mu, \mu-\alpha, \mu-2\alpha, \cdots, \mu-m\alpha.$$

Consider the action of S_α on

$$V_\mu + V_{\mu-\alpha} + V_{\mu-2\alpha} + \cdots + V_{\mu-m\alpha}.$$

This is a direct sum of simple S_α-modules with highest weight m', with $m' \le m$.

1.32 Day 32 - 11/16/12

Continuing from last class, we defined

$$c_L = \sum_{i=1}^{\ell} h_i k_i + \sum_{\alpha\in\Phi} x_\alpha z_\alpha.$$

c_L acts on $V(\lambda) = V$ as some scalar c. We want to compute

$$\mathrm{tr}_{V_\nu}(c_L) = cm(\nu).$$

Well,

$$\begin{aligned}
\mathrm{tr}_{V_\nu}(c_L) &= \sum i = 1^\ell \,\mathrm{tr}_{V_\nu}(h_i k_i) + \sum_{\alpha\in\Phi} \mathrm{tr}_{V_\nu}(x_\alpha z_\alpha)\\
&= m(\mu)(\mu,\mu) + \sum_{\alpha\in\Phi} \mathrm{tr}_{V_\nu}(x_\alpha z_\alpha).
\end{aligned}$$

Fix α. Let $\mu = \nu + k\alpha \in \Lambda^+ \cap \Pi(\lambda)$ such that $\mu + \alpha \notin \Pi(\lambda)$. Consider the subspace

$$\sum_{k\ge 0} V_{\mu-k\alpha}.$$

This is an S_α-module and so it is a direct sum of simple S_α-modules. Let $m := (\mu, \alpha^\vee)$. m is the highest S_α-weight occuring.

In a simple S_α-module with highest weight $m-2i$, the basis vector corresponding to weight $\mu - k\alpha$ is the vector v_{k-i} of the module with highest weight $m-2i$. We find that

$$x_\alpha z_\alpha v_{k-i} = \left((m-i-k)(k-i+1)\frac{(\alpha,\alpha)}{2}\right) v_{k-i}.$$

After some algebra, if we assume that $0 \le k \le m/2$, then

$$\operatorname{tr}_{V\frac{\mu - k\alpha}{\nu}}(x_\alpha z_\alpha) = \sum_{i=0}^{k} m(\mu - i\alpha)(\mu - i\alpha, \alpha).$$

Lemma: For any $\omega \in \Lambda$,

$$\sum_{i=-\infty}^{\infty} m(\omega + i\alpha)(\omega + i\alpha, \alpha) = 0.$$

Now let $\mu \in \Pi(\lambda)$ be our previous ν. By the lemma,

$$\sum_{i=1}^{\infty} m(\mu - i\alpha)(\mu - i\alpha, -\alpha) = m(\mu)(\mu,\alpha) + \sum_{i=1}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha).$$

Hence,

$$\begin{aligned}
\operatorname{tr}_{V_\mu}(c_L) &= \frac{(\mu,\mu)m(\mu)}{\sum_{i=0}^{\ell} h_i k_I} + \sum_{\alpha\in\Phi}\sum_{i=0}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha) \\
&= (\mu,\mu)m(\mu) + \sum_{\alpha>0} m(\mu)(\mu,\alpha) + 2\sum_{i=0}^{\infty} m(\mu + i\alpha)(\mu + i\alpha, \alpha).
\end{aligned}$$

This completes the proof of **Freudenthal's Formula**. □

1.32.1 Formal Characters

Notation: Consider the group ring $\mathbb{Z}[\Lambda]$. Since Λ is an additive group, to avoid confusion, we take as a basis for the group ring the set

$$\{e(\lambda) \mid \lambda \in \Lambda\}$$

with

$$e(\lambda)e(\mu) = e(\lambda + \mu).$$

Definition: If V is a finite dimensional module, and for $\mu \in \Lambda$ we have $m(\mu) = \dim(V_\mu)$, then the formal character of V is

$$\operatorname{Ch}(V) = \sum_{\mu} m(\mu)e(\mu) \in \mathbb{Z}[\Lambda].$$

Lemma: $\operatorname{Ch}(V \otimes_F W) = \operatorname{Ch}(V)\operatorname{Ch}(W)$.

1.33 Day 33 - 11/19/12

Recall: $\mathbb{Z}[\Lambda]$ is the group ring of Λ, the group of integral weights. $\mathbb{Z}[\Lambda]$ has basis $\{e(\lambda) \mid \lambda \in \Lambda\}$. Since Λ is also additive, we convert the group addition in Λ into multiplicative notation using

$$e(\lambda + \mu) = e(\lambda)e(\mu).$$

Definition: If $V(\lambda)$ is the simple L-module with highest weight $\lambda \in \Lambda^+$, then

$$\mathrm{Ch}_\lambda = \sum_{\mu \in \Pi(\lambda)} m(\mu)e(\mu) \in \mathbb{Z}[\Lambda]$$

is the formal character. Note that $m(\mu) = \dim(V(\lambda)_\mu)$.

Definition: For any $\mu \in \Lambda^*$, let Sym() be he sum of all $e(\nu)$, where ν is a W-conjugate of μ. Note that $\mathrm{Sym}(\mu) \in \mathbb{Z}[\Lambda]$.

We know that $\Pi(\lambda)$ is a saturated set of weights with highest weight λ. So, $\Pi(\lambda)$ is a union of W-orbits of weights, such that the unique dominant weight in each orbit is $\leq \lambda$. So we see that

$$\begin{aligned}\mathrm{Ch}_\lambda &= \sum \mu \in \Lambda m(\mu)e(\mu) \\ &= \sum_{\substack{\mu \in \Lambda^+ \\ \mu < \lambda}} m(\mu)\,\mathrm{Sym}(\mu)\end{aligned}$$

and $m(\mu) \neq 0$ for all $\mu \in \Lambda^+$ and $\mu < \lambda$.

1.33.1 Weyl's Character Formula

Definition: For $\sigma \in W$, define $\mathrm{sgn}(\sigma) = \det_E(\sigma) = (-1)^{\ell(\sigma)}$.

Definition: For $\gamma \in \Lambda^+$, define

$$w(\gamma) = \sum_{\sigma \in W} \mathrm{sgn}(\sigma)e(\sigma(\gamma)) = \left(\sum_{\sigma \in W} \mathrm{sgn}(\sigma)\sigma\right)(e(\gamma)).$$

Weyl's Character Formula: Let $\delta := \frac{1}{2}\sum_{\alpha > 0} \alpha$. Let $\lambda \in \Lambda^+$. Then,

$$w(\delta)\,\mathrm{Ch}_\lambda = w(\lambda + \delta).$$

Corollary:

$$\dim(V(\lambda)) = \frac{\prod_{\alpha>0}(\lambda + \delta, \alpha^\vee)}{\prod_{\alpha>0}(\delta, \alpha^\vee)}.$$

w

1.34 Day 34 - 11/26/12

1.34.1 Invariant Polynomial Functions

Let V be a finite dimensional vector space. Consider

$$S(V^*) \equiv P(V),$$

the ring of polynomial *functions* on V. (When we consider functions rather than formal polynomials, we allow two different polynomials to be equal as functions. For example $x = x^2$ over $\mathbb{F}_2$.)

If H is a Cartan subalgebra, then $P(H)$ is generated as an algebra by $\{\lambda^k \mid \lambda \in \Lambda,\ k \in \mathbb{Z}^+\}$.

Definition: W acts on H, H^*, and $P(H)$. Let $P(H)^W$ be the subalgebra of functions fixed by W.

For example, if $L = \mathfrak{gl}(n, F)$ and H is the subalgebra of diagonal matrices and $P(H) = F[x_1, \ldots, x_n]$ where $x_i(h) = i^{\text{th}}$ diagonal entry of h and $W \cong S_n$ permutes the variables, then

$$P(H)^W = \text{ the algebra of symmetric polynomials.}$$

Definition: For $f \in P(H)$ let $\text{sym}(f) \in P(H)^W$ be the orbit sum of W-orbits of f. Then,

$$\{\text{sym}(\lambda^k) \mid \lambda \in \Lambda^+,\ k \in \mathbb{Z}^+\}$$

spans $P(H)^W$.

Remark: Let L be a semisimple Lie algebra. Let

$$G = \text{Int}(L) = \langle \exp(\text{ad}\, x) = 1 + \text{ad}\, x + \frac{(\text{ad}\, x)^2}{2!} + \cdots \mid x \text{ is nilpotent}\rangle \subseteq \text{Aut}(L).$$

G acts on L and hence on L^* by $(\sigma f)(x) = f(\sigma^{-1}(x))$. Hence it also acts on $P(L) = S(L^*)$. Consider $P(L)^G$, the subalgebra of G-invariant polynomial functions.

Example: Let $L = \mathfrak{Sl}(n, f)$. Then,

$$(\exp(\text{ad}\, x))(y) = (\exp(x))y(\exp(x))^{-1}.$$

So, $G = \text{Int}(L)$ acts as conjugation by some invertible matrices. Thus,

$$\text{tr}(y^k) = \text{tr}((\sigma^{-1}y)^k)$$

for all $\sigma \in G$. So, the function $y \mapsto \text{tr}(y^k)$ is an invariant polynomial function.

Remark: More generally, if $\phi : L \to \mathfrak{gl}(V)$ is irreducible, then the function $x \mapsto \text{tr}(\phi(x)^k)$ lies in $P(L)^G$.

If $h \in H$, then

$$\phi(h) \sim \begin{pmatrix} -u_1(h) & & & \\ & u_2(h) & & \\ & & \ddots & \\ & & & u_r(h) \end{pmatrix},$$

$$\phi(h)^k \sim \begin{pmatrix} u_1(h)^k & & & \\ & u_2(h)^k & & \\ & & \ddots & \\ & & & u_r(h)^k \end{pmatrix}.$$

The u_i are the weights of the representation. So, $f\big|_H$ is a sum of λ^k for $\lambda \in \Lambda$. Consider the restriction map

$$P(L) \to P(H)$$

defined by

$$f \mapsto f\big|_H.$$

We have $P(L)^G \subseteq P(L)$ and $P(H)^W \subseteq P(H)$. We claim that if $f \in P(L)^G$, then $f\big|_H \in P(H)^W$. Additionally, we claim (**Chevalley's Theorem**) that the map $\theta : P(L)^G \to P(H)^W$ is surjective.

> **Proof of second claim:** Recall that $P(H)^W$ is generated by $\mathrm{sym}(y^k)$ for $\lambda \in \Lambda^+$. So, it suffices to prove that for all $\lambda \in \Lambda^+$ and $k \in \mathbb{Z}$, there exists $f \in P(L)^G$ such that $f\big|_H = \mathrm{sym}(y^k)$. Use induction on Λ^+ with the partial order $\prec$. □

1.35 Day 35 - 12/03/12

Definition: Define $p(\lambda)$ to be the number of sets $\{k_\alpha\}_{\alpha>0}$ of nonnegative integers such that $\sum_\alpha k_\alpha \alpha = -\lambda$. As functions $p = \mathrm{ch}_{Z(0)}$ and $p(\mu - \lambda) = \dim(Z(\lambda)_\mu)$.

Definition: Define

$$\begin{aligned} q &:= \prod_{\alpha>0} (\epsilon_{\alpha/2} - \epsilon_{-\alpha/2}) \\ &= \epsilon_\delta * \prod_{\alpha>0} (\epsilon_0 - \epsilon_{-\alpha}) \\ &= \epsilon_{-\delta} * \prod_{\alpha>0} (\epsilon_\alpha - \epsilon_0). \end{aligned}$$

Definition: The Weyl function is defined for any $\alpha > 0$ by

$$f_\alpha(\nu) := \begin{cases} 1, & \text{if } \nu = -k\alpha, k \in \mathbb{Z}^+ \\ 0, & \text{otherwise} \end{cases}.$$

Really,

$$f_\alpha = \text{“}\epsilon_0 + \epsilon_{-\alpha} + \epsilon_\alpha + \cdots\text{''}.$$

Lemma A:

(a) $p = \prod_{\alpha>0} f_\alpha$.

(b) $(\epsilon_0 - \epsilon_{-\alpha}) * f_\alpha = \epsilon_0$.

(c) $q = \epsilon_0 * \prod_{\alpha>0} (\epsilon_0 - \epsilon_{-\alpha})$.

Lemma B: Let $\sigma \in W$. Then, $\sigma q = (-1)^{\ell(\sigma) - \mathrm{sgn}(\sigma)} q$.

> **Proof:** It suffices to prove for σ_α a simple reflection. Well, σ_α maps $\alpha \mapsto -\alpha$, and permutes all other positive roots. □

Lemma C: $q * p * \epsilon_{-\delta} = \epsilon_0$.

Lemma D: $\mathrm{ch}_{Z(\lambda)}(\mu) = p(\mu - \lambda) = (p * \epsilon_\lambda)(\mu)$.

Lemma E: $q * \mathrm{ch}_{Z(\lambda)} = \epsilon_{\lambda+\delta}$.

Definition: Let M_λ be the collection of L-modules V such that

(1) V is a direct sum of weight spaces,

(2) ζ acts on V by scalars $X_\lambda(z)$, $z \in \zeta$,

(3) The formal character of V belongs to χ.

Lemma: Let $V \in M_\lambda$. Then, V has a maximal vector.

Proof: (sketch) For $\lambda \in H^*$, set

$$\theta(\lambda) := \{\mu \in H^* \mid \mu \prec \lambda \text{ and } \mu \sim \lambda\}.$$

If $\mu \in \theta(\lambda)$, then $\theta(\mu) \leq \theta(\lambda)$. Then there exists σ such that $\sigma(\mu + \delta) = (\lambda + \delta)$.

Proposition: Let $\lambda \in H^*$.

(a) $Z(\lambda)$ has a (finite) composiution series.

(b) Each composition factor of $Z(\lambda)$ is of the form $V(\mu)$, for $\mu \in \theta(\lambda)$.

(c) $V(\lambda)$ occurs only once as a composition factor of $Z(\lambda)$.

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