MAD 6207 - Graph Theory

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Contents

This packet consists mainly of notes, homework assignments, and exams from MAD6207 Graph Theory taught during the Spring 2013 semester at the University of Florida. The course was taught by Prof. V. Vatter. The notes for the course follow *Graph Theory*, by Reinhard Diestel. Numbering in these notes corresponds to the numbering in the text.

If you find any errors or you have any suggestions, please contact me at jay.pantone@gmail.com.

Chapter 1

The Basics

1.1 Definitions and Notation

Definition: A graph G is an ordered pair of sets $G = (V, E)$, where V is called the vertex set and E is called the edge set, with the property that $E \subseteq [V]^2$ (where $[V]^2$ is the set of 2-subsets of V).

Definition: Given a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. If $v \in V(G)$ or $e \in E(G)$, we use the short hand $v \in G$ or $e \in G$.

Definition: A trivial graph is one which has either 0 vertices or 1 vertex.

Definition: If the vertex v lies in the edge e, we sat that v is incident with e.

Definition: We say that the edge $e = \{v, w\}$ joins v and w. We use the shorthand $e = vw$, and we say that v and w are adjacent.

Definition: We say that $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a bijection $\phi : V \to V'$ which preserves edges, i.e.,

$$[vw \in E] \Longleftrightarrow [\phi(v)\phi(w) \in E'].$$

Definition: A set of graphs closed under isomorphism is a graph property.

Definition: A graph G is a subgraph of G' if $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$. This is a partial order on graphs.

Example: Every graph on n vertices is a subgraph of K^n, the complete graph of order n.

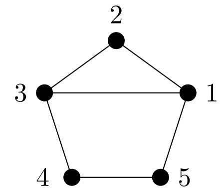
Definition: G is an induced subgraph of G' if $V(G) \subseteq V(G')$ and

$$E(G) = E(G') \cap [V(G)]^2.$$

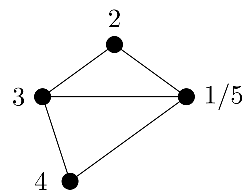
So, we get an induced graph by deleting vertices, and then only deleting edges if the were incident with a vertex which was deleted..

Definition: To contract an edge, we combine the two adjacent vertices incident with an edge e, keeping all edges which were incident with either of the vertices.

Example: The graph



is contracted to



when we combine vertices 1 and 5.

Definition: G is a minor of G' if G can be obtained from G' by:

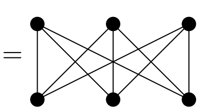
(i) deleting vertices,

(ii) deleting edges,

(iii) contracting edges.

Fact: Given a surface S, the set of graphs that can be drawn on S without edge crossings is closed downward under the minor ordering. We will prove this later.

Kuratowski's Theorem: G can be drawn on the plane if and only if G does not contain K^5 or

$K^{3,3} =$ 

We will prove this later.

Minor Theorem: (Robertson-Seymour, 1980 – 2000) In any infinite collection of graphs, one is a minor of another. This shows that given any surface, the number of minimal minors which need to be checked is finite. This theorem spans 20 papers and about 500 pages. A sketch will likely be given at the end of the course.

1.2 Degrees

Definition: The degree $d_G(v)$ of v is the number of edges incident with v. We abbreviate to $d(v)$ when the context is clear. We also define

$$\delta(G) = \min\{d(v) \mid v \in G\},$$

$$\Delta(G) = \max\{d(v) \mid v \in G\},$$

$$d(G) = \frac{1}{|V|} \sum_{v \in G} d(v) = 2 \cdot \frac{|E|}{|V|},$$

$$\epsilon(G) = \frac{|E|}{|V|} = \frac{1}{2} d(G).$$

Note that

$$\delta(G) \leq d(G) \leq \Delta(G).$$

Proposition 1.2.1: The number of vertices of odd degree is always even.

Proof: $|E| = \frac{1}{2} \sum_{v \in G} d(v)$, and so $\sum_{v \in G} d(v)$ must be even. □

Proposition 1.2.2: Every graph G with at least one edge has an induced subgraph H with

$$\delta(H) \geq \epsilon(G) = \frac{1}{2} d(G).$$

Proof: We start by deleting vertices of low degree. Make a sequence

$$G = G_0 \supseteq G_1 \supseteq \cdots$$

defined by: if G_i has a vertex v_i of degree $d_{G_i}(v_i) \leq \epsilon(G_i)$, then G_{i+1} is the induced subgraph $G_i \setminus v_i$. Now, see that $\epsilon(G_{i+1}) \geq \epsilon(G_i)$ by construction.This can't go on forever. Thus, at the end, we have a proper subgraph which meets the criterion. □

1.3 Paths & Cycles

Definition: A path is a graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\}$$
$$E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}.$$

We say that x_0 and x_k are linked by P, that x_0 and x_k are the end vertices of P, and that $x_1, \ldots, x_{k-1}$ are the inner vertices of P.

According to our textbook (Diestel), the length of a path is the number edges. (Many other sources define the length to be the number of vertices. We will stick to Diestel's convention.)

"The" path (up to isomorphism) of length k is denoted by P^k.

Notation: Given a path $P = x_0 \cdots x_k$, we use the following notations:

$$Px_i := x_0 \cdots x_i,$$
$$x_iP := x_i \cdots x_k,$$
$$x_iPx_j := x_i \cdots x_j,$$
$$\mathring{P} = x_1 \cdots x_{k-1}.$$

Definition: Given a path $P = x_0x_1 \cdots x_{k-1}$, the graph

$$P + x_{k-1}x_0$$

is a cycle, and it can be denoted $x_0x_1 \cdots x_{k-1}x_0$. We denote this cycle by C^k.

Definition: Given a graph G, its girth, denoted $g(G)$, is the minimum length of a cycle in G, and its circumference is the maximum length of a cycle in G.

Remark: Observe that if you make the minimum degree $\delta(G)$ big (by connecting everything to get a complete graph), then $g(G)$ is 3. So, a natural question is: Can you have large $\delta(G)$ and large $g(G)$ simultaneously? Erdös proved non-constructively that the answer is yes. A random graph with large $\delta(G)$ will also have large $g(G)$.

Definition: A chord is an edge between two vertices of a cycle that is not in the cycle. An induced cycle is a cycle which has no chords.

Proposition 1.3.1: Every graph contains a path of length $\geq \delta(G)$ and a cycle of length $\geq \delta(G)+1$ (assuming $\delta(G) \geq 2$).

Proof: Let $x_0 x_1 \cdots x_k$ be a path of maximum length in G. We want to show that $k \geq \delta(G)$. Since the length is maximum, all of x_k's possible neighbors must be among $x_0, \ldots, x_{k-1}$. Thus, $k \geq \delta(G)$.

Now, let $i < k$ be minimal such that $x_i \sim x_k$ (i.e., x_i is the first vertex along the path which is adjacent to x_k). Then the cycle $x_i \cdots x_k x_i$ must contain all neighbors of x_k, and so it contains more than $\delta(G)+1$ vertices, and thus has length $\geq \delta(G)+1$. □

Definition: The distance between two vertices x and y, denoted $d_G(x,y)$, is the length of the shortest path the links them.

Definition: The diameter of a graph, denoted $\operatorname{diam}(G)$, is the greatest distance between the two vertices of G.

Proposition 1.3.2: If G contains a cycle, then $g(G) \leq 2\operatorname{diam}(G)+1$.

Take a shortest cycle C. If C has length $\geq 2\operatorname{diam}(G)+2$, then we can find a shorter path by getting to $x_{\operatorname{diam}(G)+1}$ and then go back by a path of length $\operatorname{diam}(G)$ guaranteed by the definition of $\operatorname{diam}(G)$. If the path hits the cycle, we have an even shorter cycle. □

Definition: A vertex if called central if its greatest distance to another vertex is as small as possible. This distance is called the radius and is denoted $\operatorname{rad}(G)$. Note that

$$\operatorname{rad}(G) = \min_{x \in G} \max_{y \in G} d_G(x,y).$$

Remark: $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G)$.

Proposition 1.3.3: A graph of radius K and maximal degree $\Delta(G) = d$ has at most $1 + kd^k$ vertices.

Proof: Counting out from a central vertex in "layers", we get

$$|G| \leq 1 + d + d^2 + \cdots + d^k \leq 1 + kd^k.$$

1.4 Connectivity

Definition: A graph is connected if every two vertices are linked by a path.

Proposition 1.4.1: The vertices of a connected graph can be labelled $v_1, \ldots, v_n$ such that for all i, the induced subgraph $G[v_1, \ldots, v_i]$ is connected.

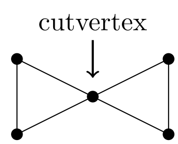
> **Proof:** Pick v_1 arbitrarily. Suppose $v_1, \ldots, v_i$ have been chosen. Pick any $x \in G \setminus G[v_1, \ldots, v_i]$. By the definition of connectedness, there is a $v_1 - x$ path. Define v_{i+1} to be the first vertex along this path which we have not yet labelled. (This exists because we at least know that x has not been labelled.) Label all vertices in this fashion. By construction $G[v_1, \ldots, v_i]$ is connected for all i. □

Technicality: The empty graph is ***not connected***.

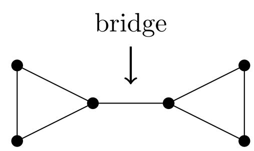
Definition: Any nonempty graph can be decomposed as the union of connected components.

Definition: Suppose $A, B \subseteq V(G)$. Let H be a subgraph of G. If all $A - B$ paths in G pass through H, then H separates A and B.

Definition: A vertex that separates two vertices of the same component is a cutvertex.



An edge that does this is called a bridge.



Definition: We say that a graph G is k-connected if $|G| > k$ and $G \setminus X$ is connected for all subsets $X \subseteq V(G)$ with $|X| \leq k - 1$. Note that if a graph is k-connected, then it is necessarily $(k-1)$-connected.

Example: K^n is $(n-1)$-connected.

Definition: The greatest k such that G is k-connected is denoted $\kappa(G)$.

Definition: The concept of edge-connectivity is analogous; we remove edges instead of vertices. The greatest k such that G is k-edge-connected is denoted $\lambda(G)$.

Example: A connected graph with a bridge is 1-edge-connected.

Proposition 1.4.2 / Homework Exercise: $\kappa(G) \le \lambda(G) \le \delta(G)$

Hint: To see the first inequality take some edges which disconnect it, and find a smaller set of vertices which also disconnects it. For the second inequality, find a vertex of minimal degree, delete all edges around it, and this disconnects it.

Remark: The exercise above shows that a graph which has large connectivity must have large minimal degree. The converges is false: connect two complete graphs with a bridge. The following theorem provides a partial converse.

Theorem 1.4.3: (Mader, 1972) If the average degree of G, $d(G)$, is at least $4k$, then G contains a k-connected subgraph.

Proof: The statement is trivial for $k = 0, 1$. We will change to weaker hypotheses:

(1) $n = |V(G)| \ge 2k - 1$,

(2) $m = |E(G)| \ge (2k-3)(n-k+1)+1$.

First we must verify that these hypotheses are indeed weaker:

(1) $n > \Delta(G) \ge d(G) \ge 4k$,

(2) $m = \frac{1}{2}d(G)n \ge 2kn$.

We proceed by induction on n. If $n = 2k-1$ then $k = \frac{1}{2}(n+1)$. Thus, by (2),

$$\begin{aligned} m &\ge (2k-3)(n-k+1)+1 \\ &= (n+1-3)\left(n - \frac{1}{2}n - \frac{1}{2} + 1\right) + 1 \\ &= (n-1)\left(\frac{1}{2}n + \frac{1}{2}\right) + 1 \\ &= (n-2)\frac{1}{2}(n+1) + 1 \\ &\ge \frac{1}{2}n(n-1) \\ &= \binom{n}{2}. \end{aligned}$$

The only way to have at least $\binom{n}{2}$ edges is if $G = K^n \supseteq K^{k+1}$, and K^{k+1} is k-connected.

Now suppose that $n \ge 2k$. If $v \in G$ has $d(v) \le 2k-3$, then we're done by induction. So, suppose that $\delta(G) \ge 2k-2$. If G is connected, then we're done. Thus, suppose that G is composed of nonempty subgraphs G_1 and G_2 such that

$$|G_1 \cap G_2| \le k-1$$

and there are no edges between G_1 and G_2 which are not incident with the vertices in $G_1 \cap G_2$. By the condition on $\delta(G)$, we know that

$$|G_i| \ge 2k-1$$

for $i = 1, 2$. If G_1 or G_2 satisfy (1) and (2), then we're done by induction because

$$\|G_i\| \le (2k-3)(|G_i| - k + 1)$$

($\|G\|$ denotes the number of edges of G). Therefore,

$$\begin{aligned} m &= \|G_1\| + \|G_2\| \\ &\le (2k-3)(|G_1| + |G_2| - 2k + 2). \end{aligned}$$

Note that $|G_1| + |G_2| \leq n + k - 1$, and so

$$m \leq (2k-3)(n-k+1),$$

a contradiction. □

1.5 Trees

Definition: An acyclic graph is called a forest. A connected forest is called a tree.

Theorem 1.5.1: The following are equivalent for a graph T.

(1) T is a tree.

(2) Any two vertices of T are linked by a unique path.

(3) T is minimally connected, i.e., $T - e$ is disconnected for all edges $e \in T$.

(4) T is maximally acyclic, i.e., $T + xy$ is not acyclic for all non-adjacent $x, y \in T$.

Proof:

(1) $\Longrightarrow$ (2): If there are two such paths, then there is some cycle.

(2) $\Longrightarrow$ (3): Let $e = xy$. e is the unique path from x to y, so there can be no other such path in $T - e$.

(3) $\Longrightarrow$ (1): We have connectedness. If the graph had a cycle, then we could remove something and still be connected. Therefore the graph is acyclic.

(2) $\Longrightarrow$ (4): Adding an edge would create two paths, hence a cycle.

(4) $\Longrightarrow$ (2): By maximality, T is connected. So, there exists a path. If there are two paths, then the graph has a cycle, which can't be true. □

Notation: Denote by xTy the unique path from x to y in a tree T.

Theorem: Every connected graph contains a spanning subtree, which has all of the same vertices but is a tree.

Proof: Delete edges wherever you can to leave the graph connected. The result is minimally connected, and hence a tree. □

Corollary 1.5.3: A connected graph on n vertices is a tree if and only if it has $n - 1$ edges.

Proof: (by induction, separately in each direction) In the base case, the graph has 1 vertex and 0 edges, and is trivially a tree.

($\Longrightarrow$) If you take a tree and add a vertex, you must add exactly one edge, since adding two edges would make a cycle. □

($\Longleftarrow$) Since the graph is connected, it has a spanning subtree which must have $n - 1$ edges by the previous direction. So, this tree is the whole thing. □

Corollary 1.5.4: Let T be a tree and let G be any graph. If $\delta(G) \geq |T| - 1$, then G contains T as a subgraph.

> **Proof:** Use **Corollary 1.5.2** to get a labeling. Then, pick any vertex in G as v_1, and any neighbor as v_2, etc. Since $\delta(G) \geq |T| - 1$, we'll never run out of neighbors. □

Definition: Sometimes we mark one vertex of a tree as a root. We call such a tree a rooted tree. Let r be the root. We get an induced partial order of vertices:

$$[x \leq y] \iff [x \in rTy],$$

i.e., x is on the unique path between the root and y.

1.6 Bipartite Graphs

Definition: A graph G is r-partite if its vertices can be partitioned into r nonempty subsets such that there are no edges between two vertices in the same subset. A graph is said to be bipartite if it is 2-partite.

Definition: A complete r-partite graph is an r-partite graph which has all possible edges.

Definition: Let G and H be graphs. Then, $G * H$ is defined by

$$\begin{aligned} V(G * H) &= V(G) \cup V(H) \\ E(G * H) &= E(G) \cup E(H) \cup \{xy \mid x \in G, y \in H\}. \end{aligned}$$

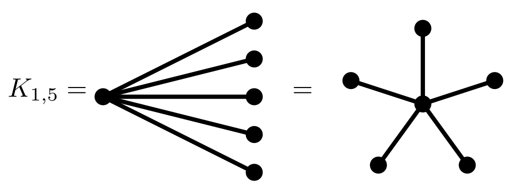
Remark: So, a complete r-partite graph can be expressed as

$$K_{n_1, n_2, \ldots, n_r} := \overline{K^{n_1}} * \overline{K^{n_2}} * \cdots * \overline{K^{n_r}},$$

where $\overline{K^{n_1}}$ is the graph which has n_1 vertices and no edges. This is now the unique complete r-partite graph with parts of size $n_1, n_2, \ldots, n_r$.

Notation: $K_s^r := K_{\underbrace{s, s, \ldots, s}_{r \text{ parts}}}$.

Definition: $K_{1,n}$ is called a star.



Theorem 1.6.1: [[***Could be on Quals!***]] The graph G is bipartite if and only if it has no odd cycles.

> **Proof:**
>
> ($\Longrightarrow$) (by contrapositive) It's clear that if the graph has an odd cycle, then it cannot be bipartite because there must be an edge between two vertices in the same part. □

($\Longleftarrow$) Suppose G does not contain an odd cycle. It suffices to consider the case where G is connected. Let T be a spanning tree of G. Root T at the vertex r. Define

$$A = \{v \mid rTv \text{ has even length}\},$$
$$B = \{v \mid rTv \text{ has odd length}\}.$$

We want A and B to form the parts of a bipartite graph. So, we want each edge to have one vertex in each set. Suppose $e = xy \in G$. If $e \in T$, then we must immediately have that $x \in A$ and $y \in B$ or vice versa. If $e \notin T$, then $T + e$ must have a cycle. By assumption, this cycle has even length. So, xTy has an odd number of edges. So, either rTx is odd and rTy is even, or vice versa. Therefore, $x \in A$ and $y \in B$, or vice versa. This shows that A and B are the parts in our bipartite graph. □

1.7 Minors

Definition: Recall the definition of edge contraction from section 1.1. If we contract the edge $e = xy$, then we use the notation G/e, defined by

$$V(G/e) = V(G) - \{x, y\} \cup \{v_e\}$$
$$E(G/e) = E(G) \setminus \{f \mid f \text{ is incident to } x \text{ or } y\} \cup \{v_e z \mid z \text{ is adjacent to } x \text{ or } y\}.$$

Definition: Recall that if G can be obtained from H by deleting vertices, deleting edges, or contracting edges, then we say that G is a minor of H.

Definition: If X is a graph and $\{V_x \mid x \in X\}$ is a partition of G into connected subgraphs, and if

$$[xy \in E(x)] \Longleftrightarrow [\exists xy \in G, \text{ with } x \in V_x \text{ and } y \in V_y],$$

then we say that G is an MX. (X is the graph obtained by contracting edges, and the "M" stands loosely for "minor" − loosely, because we have not allowed deletion of edges or vertices.)

Fact: G is an MX if and only if we can obtain X by contracting edges of G.

Definition: Diestel defines a minor with the following definition: If G is an MX and G is a subgraph of Y, then X is a minor of Y.

Definition: Let X be a graph. If G can be obtained by expanding edges of X with independent paths, then G is a subdivision of X, and we say that G is a TX. If G is a TX and G is a subgraph of Y, then X is a topological minor.

Chapter 2

Matching, Covering, and Packing

2.1 Matching in Bipartite Graphs

Definition: A set of independent edges is called at matching.

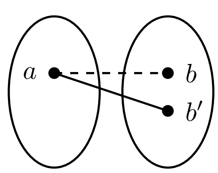
Definition: A k-regular spanning subgraph is called a k-factor.

Remark: Consider a bipartite graph with parts A and B. An alternating path starts in A at an unmatched vertex and alternates between edges in our matching M and edges in $E \setminus M$. An augmenting path is an alternating path which ends at an unmatched vertex.

König's Theorem (1931) For a bipartite graph, the size of the smallest cover is equal to the size of the largest matching.

Proof: Take a maximal matching M. For every edge in M, choose an end; its end is in B if an alternating path ends there, and its end is in A otherwise. Let U denote this set of $|M|$ vertices. Take $ab \in E$. We want $a \in U$ or $b \in U$. So, assume that $ab \notin M$. Also, by maximality of M, we see that M has an edge $a'b'$ with $a = a'$ or $b = b'$.

If a is unmatched in M, then $b' = b$. But then ab is an alternating path, so $b \in U$ and ab is covered. So, suppose $a = a'$, as in the picture below:



If $a \in U$, then we're done. So, assume $a \notin U$. Since $ab' \in M$, we have that $b' \in U$. So, there is an alternating path P ending at b'. P can be extended (or shrunk) to end at b. If b is unmatched by M, then form an augmenting path, yielding a contradiction. If b is matched by M¡ then this alternating path puts $b \in U$. □

Hall's Marriage Theorem: (1935) The bipartite graph G with parts A and B contains a matching go A (i.e., every $a \in A$ is matched) if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

Proof: The given condition is clearly necessary. To see sufficiency, suppose that G has no matching of A. By **König's Theorem**, G has a cover U with $|U| < |A|$. Say that $U = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$. Then,

$$|A'| + |B'| = |U| < |A|.$$

Thus,

$$|B'| < |A| - |A'|$$

and so

$$B'| < |A \setminus A'|.$$

Remember that G has no edges between $A \setminus A'$ and $B \setminus B'$. But,

$$|N(A \setminus A')| < |B'|$$

and so

$$|N(A \setminus A') < |A \setminus A'|.$$

This is a contradiction. □

2.2 Matching in General Graphs

Definition: Define $\underline{q(G)}$ to be the number of odd components of G.

Remark: The following condition is necessary for G to have a matching:

$$q(G \setminus S) \leq |S|$$

for all $S \subseteq V(G)$. Setting $S = \emptyset$ shows us that $|V(G)|$ must be even.

Tutte's Theorem: (1947) The graph G has a 1-factor if and only if $q(G - S) \leq |S|$ for all $S \subseteq V(G)$.

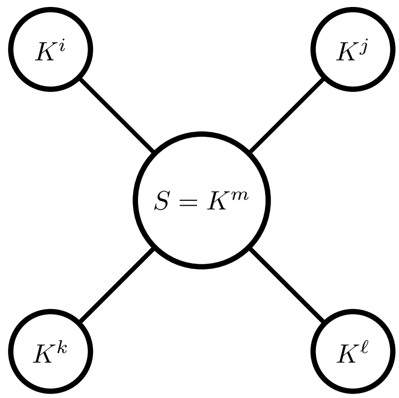
Remark: The main use of Tutte's Theorem is as a "certificate for unmatchability", because to test for matchability, we have to check 2^n subsets.

Proof: We have already observed the ($\Longrightarrow$) direction. We now show the ($\Longleftarrow$) direction by contrapositive. Let $G = (V, E)$ be a graph without a 1-factor. We want to find a "bad set" $S \subseteq V$ with $q(G - S) > |S|$.

We claim that we may take G to be edge-maximal without a 1-factor.

Proof of Claim: Suppose G' is obtained by adding edges to a graph G and that G' has a bad set S. Every odd component of $G' - S$ contains an odd component of $G - S$. So, S is a bad set for G, too.

So, if G has a bad set S, we have the following situation:



and we still have no matching. $(*)$ In the graph above, all components of $G - S$ are connected and every $s \in S$ is adjacent to all other vertices.

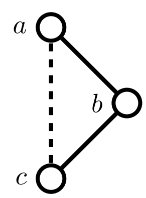
Assume that G doesn't have a 1-factor. Then, we claim that if S is as in the graph above, then G has a bad set (which is not necessarily S).

> **Proof of Claim:** If S isn't bad, then by $(*)$, we can match all of the vertices (which would be a contradiction) unless $|V|$ is off, in which case $\emptyset$ is a bad set.

Set

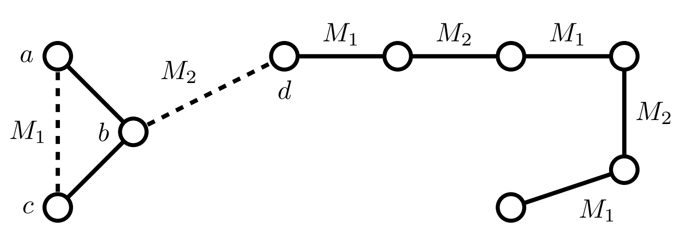
$$S := \{v \in V \mid v \text{ is adjacent to all of } V - v\}.$$

If S satisfies $(*)$, then we're done. So, if S does not satisfy $(*)$, then some component of $G - S$ is not complete, i.e. has nonadjacent vertices a and a'. Let a shortest aPa' path in $G - S$ begin abc. (Note that c might be a'. That's fine.) We have the situation below.



Note that $b \notin S$. Therefore, there is some d such that $bd \notin E$.

By edge maximality, we may construct matchings M_1 of $G + ac$ and M_2 of $G + bd$.



Take a maximal part starting at d that alternates between M_2 edges and M_1 edges. Suppose it ends at v. If we stop at an M_1 edges, then $v = b$. If we stop at an M_2 edge, then $v \in \{a, c\}$.

Let P be this path. If $v = b$, set $C = P + bd$. If $v \in \{a, c\}$, set $C = P + vb + bd$. Either way C has even length, and every second edge is in M_2. The only edge of C not in G is bd.

Take M_2, and replace its edges along C with the M_1 edges along C (along with possibly vb - but this is an M_1 edge). This gives a matching of G, which is a contradiction. □

Petersen's Theorem: (1891) Every bridgeless cubic graph ("cubic" means every vertex has degree 3) has a 1-factor.

Proof: (via Tutte's Theorem) Let $S \subseteq V(G)$. If there are no odd components of $G - S$, we're done. Otherwise, choose an odd component C of $G - S$. Note that

$$\sum_{v \in C} \deg_G(v) = 3|C|$$

is odd. Similarly,

$$\sum_{v \in C} \deg_{G[C](v)} = 2 \cdot (\# \text{ of edges in } G[C]),$$

which is even. So, there are an odd number of edges which go from C to S. Could there be only one of these edges? No, because then this edge would be a bridge. Hence there are at least 3 of these edges. This is true for each C we could have picked. Hence,

$$\left[\# \text{ edges in } G \text{ from } S \text{ to } G - S\right] \geq 3q(G - S).$$

But, since G is cubic,

$$\left[\# \text{ edges in } G \text{ from } S \text{ to } G - S\right] \leq 3|S|.$$

Hence $q(G - S) \leq |S|$. Therefore, by **Tutte's Theorem**, G has a 1-factor. $\square$

Chapter 3

Connectivity

Recall: G is <u>k-connected</u> if $G - X$ is connected (and nonempty) for all $X \subseteq V(G)$ with $|X| < k$.

Remark: This definition of k-connectedness does not help much in proofs. A better formulation is given by the following theorem.

Menger's Theorem: (1927) If G is k-connected, then every two vertices in G can be linked by k independent paths. We will prove this (in fact, a stronger version) later.

3.1 2-Connected Graphs and Subgraphs

Whitney's Ear Decomposition: (1932) The graph G is 2-connected if and only if there is a sequence $G_0, G_1, \ldots, G_\ell = G$ where G_0 is a cycle and G_{i+1} is obtained from G_i by adding a G_i-path (a path whose two end vertices are in G_i).

> **Proof:** It's clear that if a graph satisfies this sequence property, then it is 2-connected, since (loosely) anytime we remove a vertex, we're always left with two halves of a cycle. It remains to show that every 2-connected graph has such a sequence.
>
> Let G be 2-connected. Then, G contains a (nontrivial) cycle (see **Homework 1**) G_0. Therefore, G has a nontrivial maximum subgraph H which has an ear decomposition (at worst, it's just the cycle G_0).
>
> We claim that H is an induced subgraph. If it were not induced, there would be vertices $x, y \in V(H)$ such that $xy \in E(G)$, but $xy \notin E(H)$. But, we can add the edge xy to H and it is still an ear decomposition, violating maximality. Hence, H is an induced subgraph.
>
> So, if $H \neq G$, then there must be a missing vertex $y \in G - H$. Since G is connected, there is some vertex $x \in H$ such that $xy \in E(G)$. Since G is 2-connected, if we remove the vertex x, then y must still be connected to $h \in H$ via some path xPh. Then, $xyPh$ is an ear in out decomposition, which again violates maximality of H. Therefore, H is missing no vertices (and since it's an induced subgraph, it's missing no edges). Therefore, $H = G$, and so G has an ear decomposition. $\square$

Block Decomposition: A maximal connected subgraph without a cutvertex is called a block. (Note: a subgraph with 1 or 2 vertices is a maximal connected subgraph without a cutvertex, but is not 2-connected.) So, a block in G is either:

(1) a maximal 2-connected subgraph,

(2) a bridge, or

(3) an isolated vertex.

Remark: By maximality, two blocks can intersect in at most one vertex: if two blocks intersected in two vertices, then we could remove one and leave the two blocks connected, which means the two vertices aren't cutvertices and so the blocks aren't maximal.

Remark: If two blocks intersect in one vertex, then that vertex is a cutvertex of G.

Remark: A cutvertex lies in at least 2 blocks.

Remark: Every edge lies in a unique block.

Remark: We can form the bipartitie block graph as follows:

$$\begin{aligned} A &= \{\text{cutvertices of } G\} \\ B &= \{\text{blocks of } G\} \\ v \sim B &\text{iff } v \in B. \end{aligned}$$

Proposition 3.1.1: If G is connected, then its block graph is a tree.

3.2 The Structure of 3-Connected Graphs

Lemma: If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is 3-connected.

Proof: Suppose otherwise that G is 3-connected, but has no such edge. Then, for every edge $xy \in G$, we see that G/xy has a separating set S (of vertices) with $|S| \leq 2$. In fact $|S| = 2$ because if $|S| = 1$ then we could have separated G with that vertex, together with either x or y, a contradiction to 3-connectedness. We must have that $S = \{v_{xy}, z\}$ where $z \in V(G) \setminus \{xy\}$.

Any two vertices separated by S in G/xy are separated by $\{x, y, z\}$ in G. Since G is 3-connected, no proper subset of $\{x, y, z\}$ separates G. Each of x, y, z must be adjacent to every component C of $G - \{x, y, z\}$.

Choose an edge xy, a vertex z, and a component C so that:

(i) G/xy is not 3-connected,

(ii) $G - \{x, y, z\}$ is disconnected,

(iii) C is a component of $G - \{x, y, z\}$,

(iv) $|C|$ is as small as possible.

Also, choose v to be a neighbor of z in C. Then, G/zv is also not 3-connected. So, there is a vertex w such that $\{v, z, w\}$ separates G.

Let D be a component of $G - \{z, v, w\}$ which contains neither x nor y. The vertex v has neighbors in D which all lie in C. Therefore, $D \subsetneq C$, a contradiction. □

Theorem: (Tutte, 1961) G is 3-connected if and only if there is a sequence $G_0, \ldots, G_n$ such that:

(1) $G_0 = K_4$ and $G_n = G$,

(2) $G_i = G_{i+1}/xy$ for an edge $xy \in G_{i+1}$ with $d(x), d(y) \geq 3$.

Proof: The ($\Longrightarrow$) direction is given by the previous lemma. (The condition on the degrees is due to the fact that if $d(x)$ or $d(y)$ is 2, then that vertex is connected to only the other of x, y and one other component, which is not possible.)

Conversely, suppose $G_0, \ldots, G_n$ is such a sequence. We want to show that if G_i is 3-connected, then G_{i+1} is also 3-connected. Let $G_i = G_{i+1}/xy$. Suppose that G_{i+1} is not 3-connected. So, there is a set S with $|S| \leq 2$ such that $G_{i+1} - S$ is disconnected. Let C_1 and C_2 be two components of $G_{i+1} - S$. Without loss of generality, let $C_1 \cap \{x, y\} = \emptyset$. If $\{x, y\} \subseteq C_2$, then S separates $G_i = G_{i+1}/xy$, a contradiction. Thus, one of x or y lies in S.

If there is a vertex $v \in C_2 \setminus \{x, y\}$, then G_i isn't 3-connected. So, without loss of generality, $x \in S$ and $C_2 = \{y\}$. This is a contradiction to out hypothesis that $d(y) \geq 3$ since at most y is adjacent to the two vertices in S. □

3.3 Menger's Theorem

Menger's Theorem - Statement 1: If G is k-connected, then there are k "edge disjoint" paths between any two of its vertices.

Menger's Theorem - Statement 2: Let $A, B \subseteq V(G)$. The minimum number of vertices separating A from B is equal to the maximum number of $A - B$ paths.

Definition: Define $K(G, A, B)$ to be the minimum number of vertices separating A from B. Note that

$$K(G, A, B) \leq \min(|A|, |B|).$$

In this definition, we do count trivial (one vertex) paths. Also, if $A \subseteq B$, then

$$K(G, A, B) = |A|$$

and if $B_1 \subseteq B_2$ then,

$$K(G, A, B_1) \leq K(G, A, B_2).$$

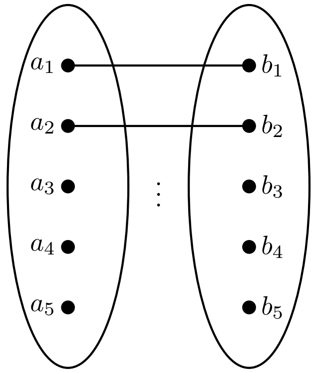
Remark: Statement 2 above is strictly stronger. To see this, let G be k-connected and let $|A|, |B| \geq k$. Then, we see that the minimum number of vertices separating A from B is at least k.

Menger's Theorem - Statement 3: Let $k = K(G, A, B)$. For $n < k$, if there are n disjoint $A - B$ paths $P_1, \ldots, P_n$, then there exists $n + 1$ (vertex) disjoint $A - B$ paths $Q_1, \ldots, Q_{n+1}$ such that every B-endpoint of a P_i is the B-endpoint of some Q_j.

Remark: Statement 3 is strictly stronger than statement 2, and more convenient to prove.

Proof: (by induction) We proceed by induction on $\beta = |G| - |B|$. In the base case $\beta = 0$ we have that $G = B$ and so $A \subseteq B$. Hence, $k = |A|$, and the rest is trivial.

Now suppose we have $n < k$ disjoint $A - B$ paths $P_1, \ldots, P_n$. Label their endpoints in A and B by a_i and b_i respectively.



Because $n < k$, the set $\{b_1, \ldots, b_n\}$ does not separate A from B. So, there is a path R from A to $B \setminus \{b_1, \ldots, b_n\}$. If R does not intersect $P_1, \ldots, P_n$, then we're done. Otherwise, let x be the last vertex of R in $P_1, \ldots, P_n$.

Relabel the paths so that $x \in P_n$. Recall the following notation: xP_n is the part of the path P_n which starts with x and P_nx is the part of the path P_n which ends with x.

Define $B' = B \cup xP_n \cup xR$. Note that $|B'| \geq |B|$ since $x \in B'$ but $x \notin B$ (otherwise x would be an endpoint). We can now apply induction, after updating the path $P_n := P_nx$, to the paths $P_1, \ldots, P_{n-1}, P_n$. Note that

$$k(G, A, B') \geq k(G, A, B)$$

because $B' \supseteq B$. By induction, there are disjoint $A - B'$ paths ${Q'}_1, \ldots, {Q'}_{n+1}$ with endpoints

$$b_1, \ldots, b_{n-1}, x, y.$$

Label these so that ${Q_i}'$ is an $A - b_i$ path, for $i = 1, \ldots, n-1$, and ${Q'}_n$ is an $A - x$ path, and ${Q'}_{n+1}$ is an $A - y$ path. Now, we need to turn these into $A - B$ paths. We have three cases, depending on where y is.

(**Case 1**) ($y \in xP_n$) Since y must be a new endpoint (by the induction), we know that $y \neq x$. Set

$$\begin{aligned} Q_i &= {Q'}_i, \text{ for } i = 1, \ldots, n-1, \\ Q_n &= {Q'}_n x \cup xR, \\ Q_{n+1} &= {Q'}_{n+1} y \cup yP_n. \end{aligned}$$

(**Case 2**) ($y \in xR$) We have the following picture. In this case, set

$$\begin{aligned} Q_i &= {Q'}_i, \text{ for } i = 1, \ldots, n-1, \\ Q_n &= {Q'}_n x \cup xP_n, \\ Q_{n+1} &= {Q'}_{n+1} y \cup yR. \end{aligned}$$

(**Case 3**) ($y \in B$) Note that $y \notin \{b_1, \ldots, b_{n-1}, x\}$. This is the good case. We take In this case, set

$$\begin{aligned} Q_i &= {Q'}_i, \text{ for } i = 1, \ldots, n-1, \\ Q_n &= {Q'}_n x \cup xP_n, \\ Q_{n+1} &= {Q'}_{n+1}. \end{aligned}$$

The proof is now complete. □

Chapter 4

Planar Graphs

4.1 Topological Prerequisites

Definition: A straight line segment is defined by

$$p + \lambda(q - p) : 0 \leq \lambda \leq 1$$

where p and q are points.

Definition: A polygon is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle.

Definition: A homeomorphism is a continuous bijection with a continuous inverse.

Definition: A polygonal arc is the union of finitely many straight line segments which is homeomorphic to $[0, 1]$.

Definition: If P is an arc between points x and y, then the interior of P is $\mathring{P} := P \setminus \{x, y\}$.

Remark: Let $\mathcal{O} \subseteq \mathbb{R}^2$ be an open set. For $x, y \in \mathcal{O}$, define $x \sim y$ if x and y can be linked by an arc in $\mathcal{O}$, or if $x = y$. This is an equivalence relation. The equivalence classes are called the regions of $\mathcal{O}$.

Definition: We say that the closed set $X \subseteq \mathbb{R}^2$ separates $\mathcal{O}$ if $\mathcal{O} \setminus X$ has more than one region.

Definition: The frontier of a set $X \subseteq \mathbb{R}^2$ is the set of all points $x \in X$ such that every neighborhood of x meets both X and $\mathbb{R}^2 \setminus X$. (The usual term for this set is boundary.)

Definition: A set is bounded if it is contained in some disc.

Jordan Curve Theorem: For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has two regions, exactly one of which is bounded. Both have P as their frontier.

Lemma 4.1.2: Let P_1, P_2, and P_3 be three arcs, all between the same endpoints but otherwise disjoint. Then,

(1) $R^2 \setminus (P_1 \cup P_2 \cup P_3)$ has three regions, with frontiers $P_1 \cup P_2$, $P_2 \cup P_3$, and $P_2 \cup P_3$.

(2) If P is an arc between a point in $\mathring{P}_1$ and a point in $\mathring{P}_3$ and if $\mathring{P}$ lies in the region of $R^2 \setminus (P_1 \cup P_2 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \emptyset$.

Lemma 4.1.3: Let $X_1, X_2 \subseteq \mathbb{R}^2$ be disjoint sets, each the union of finitely many points and arcs. Let P be an arc from a point in X_1 to a point in X_2 whose interior lies in a region of $\mathbb{R}^2 \setminus (X_1 \cup X_2)$. Then, $\mathcal{O} \setminus \mathring{P}$ is a region of $\mathbb{R}^2 \setminus (X_1 \cup X_2 \cup P)$.

Remark: Let S^n be the n-dimension (unit) sphere. Then, $S^2 \setminus (0,0,1)$ is homeomorphic to $\mathbb{R}^2$. This is called the <u>stereographic projection</u>.

Definition: Let $\pi : S^2 \setminus (0,0,1) \to \mathbb{R}^2$ be a homeomorphism. If $P \subseteq \mathbb{R}^2$ is a polygon and $\mathcal{O}$ is the bounded region of $\mathbb{R}^2 \setminus P$. Then, we make the following definitions. $C := \pi^{-1}(P)$ is a <u>circle</u> in S^2. Note that $\pi^{-1}(\mathcal{O})$ and $S^2 \setminus \pi^{-1}(P \cup \mathcal{O})$ are the <u>regions</u> of C.

Theorem 4.1.4: Let $\phi : C_1 \to C_2$ be a homomorphism between two circles on S^2 (this always exists). If $\mathcal{O}_1$ is a region of C_1 and $\mathcal{O}_2$ is a region of C_2, then ϕ can be extended to a homeomorphism

$$\overline{\phi} : C_1 \cup \mathcal{O}_1 \to C_2 \cup \mathcal{O}_2.$$

(This is not true on the plane.)

4.2 Plane Graphs

Definition: A <u>plane graph</u> is a pair (V, E) of subsets of $\mathbb{R}^2$ such that:

(1) Every edges is an arc between two elements of V.

(2) Different edges have different endpoints.

(3) The interior of any edge is disjoint of $V \cup E$.

Essentially, we want to be able to draw the graph G without any two edges crossing.

Definition: The graphs we studied earlier will be called <u>abstract graphs</u>.

Definition: The plane graph G divides $\mathbb{R}^2$ into the regions of $\mathbb{R}^2 \setminus G$. These regions are called the <u>faces</u> of G.

Remark: Every (finite) plane graph G is bounded. So, G a unique <u>outer face</u>. All other faces are <u>inner faces</u>.

Lemma 4.2.1: Let G be a plane graph with edge e.

(1) If X is the frontier of a face of G, then either $e \in X$ or $\mathring{e} \cap X = \emptyset$.

(2) If e lies on a cycle in G, then e lies on the frontier of exactly two faces of G.

(3) If e does not lie on a cycle in G, then it lies on exactly one face of G.

Corollary 4.2.2: The frontier of a face is always the point-set of a subgraph.

Proposition 4.2.3: A plane forest has exactly one face.

Lemma 4.2.4: If a plane graph has different faces with the same boundaries, then G is a cycle.

Proof: See book. Use (2) of **Lemma 4.2.1**.

Proposition 4.2.5: In a 2-connected plane graph, every face is bounded by a cycle.

Proof: See book. Use **Whitney's Ear Decomposition**.

Proposition 4.2.6: A plane graph on at least three vertices maximally planar (i.e., any edge we try to add forces a crossing, with the current drawing) if and only if it is a plane triangulation (the frontier of every face is a K^3).

Euler's Formula: For every surface S there is a constant χ such that for every connected graph G drawn on S,

$$F + V - E = \chi,$$

where F is the number of faces, V is the number of vertices, and E is the number of edges.

Theorem 4.2.7: Let G be a connected plane graph with n vertices, m edges, and ℓ faces. Then,

$$n - m + \ell = 2.$$

Proof: Use induction on m. The base case is $m = n - 1$. In this case, G is a tree, and so $\ell = 1$. Thus

$$n - m + \ell = n - (n - 1) + 1 = 2.$$

Now assume G has m edges, and that we have proved the theorem for all graphs with $< m$ edges. Let e be an edge in a cycle (which must exist because we have added at least one edge to a tree). We know that e lies on the boundary of two faces f_1 and f_2. Hence, $G - e$ has one fewer face of G. (Define

$$f_{1,2} = f_1 \cup \mathring{e} \cup f_2.$$

Then,

$$F(G) \setminus \{f_1, f_2\} = F(G - e) \setminus \{f_{1,2}\}.)$$

This completes the theorem. $\square$

Remark: Suppose we have a plane triangulation with n vertices, m edges, and ℓ faces. Every edge lies on the boundary of 2 faces. Every face has 3 edges on its boundary. So, we can double-count the set

$$(e, f) : e \text{ is on the boundary of } f.$$

We see that

$$2m = 3\ell$$

and so

$$\ell = \frac{2}{3}m.$$

Additionally, by **Euler's Formula**,

$$\begin{aligned} n - m + \ell &= 2 \\ n - m + \frac{2}{3}m &= 2 \\ n - \frac{1}{3}m &= 2 \\ 3n - m &= 6 \\ m &= 3n - 6. \end{aligned}$$

Hence we have the following corollary.

Corollary 4.2.8: A plane graph with at least 3 vertices has at most $3n - 6$ edges.

Remark: We can prove that K_5 is not planar by noting that $n = 5$ and $m = \binom{5}{2} = 10$, and so $3n - 6 = 15 - 6 = 9 \not\geq m$.

Remark: Additionally, $K_{3,3}$ is not planar, even though it does not violate the inequality in the corollary. However, we can prove that it is not planar in a different way. For bipartite graphs, every face has at least 4 edges on its boundary (since bipartite graphs can't have cycles of even length). Hence, by a similar double-counting argument, we get that

$$4\ell \leq 2m.$$

Hence,

$$4 \leq 2n - m,$$

i.e.,

$$m \leq 2n - 4.$$

$K_{3,3}$ does in violate this bound. Therefore, $K_{3,3}$ is not planar.

Recall: S is a subdivision of H if S can be obtained from H by replacing edges with independent paths. We say that H is a topological minor of G if G contains (as a (not necessarily induced) subgraph) a subdivision of H.

Remark: If G contains a K_5 or a $K_{3,3}$ as a topological minor, then G can't be drawn on the plane.

4.4 Planar Graphs: Kuratowski's Theorem

Kuratowski's Theorem: G contains a K_5 or a $K_{3,3}$ as a topological minor if and only if G is not planar.

Proposition 1.7.2 (ii): If $\Delta(X) \leq 3$, and G contains X as a minor, then G contains X as a topological minor. Hence, if G contains $K_{3,3}$ as a minor, then G also contains $K_{3,3}$ as a topological minor.

Proposition 4.4.2: If G contains a K^5 or a $K_{3,3}$ minor, then G also contains a K^5 or $K_{3,3}$ topological minor. (**Warning:** It is not true that if G contains a K_5 minor then G must also contain a K_5 topological minor.)

Proof: After considering **Proposition 1.7.2 (ii)** above, it remains to show that if G has a K^5 minor, then G also contains a K^5 or $K_{3,3}$ topological minor.

Suppose G contains a K^5 minor. Let $K \subseteq G$ be minimal such that K contains a K^5 minor. (Note, we are minimal on vertices and then edges.) We can contract along "branch sets" of K to get K^5. By minimality, each branch set is a tree and there is precisely one edge between any two branch sets.

Let V_x be the tree induced by a branch set. Extend V_x to a tree T_x by including its four neighbors (one in each other branch set).

By minimality again, T_x has precisely 4 leaves. If every T_x is a subdivision of $K_{1,4}$, then G contains K^5 as a topological minor, and so we're done. So, now assume that some T_x is not a subdivision of $K_{1,4}$. Then, the only possibility is that T_x has two vertices of degree 3. Therefore, G has a $K_{3,3}$ minor, and so the proof is complete by **Proposition 1.7.2 (ii)**. □

Lemma 4.4.3: Every 3-connected graph without a K^5 or a $K_{3,3}$ minor is planar.

Proof: We use induction on $|G|$. The base case is $G = K^4$, and its clear that the lemma is true for this G.

Now, since G is 3-connected, it has an edge so that G/xy is 3-connected. Also, G/xy does not contain K^5 or $K_{3,3}$, so by induction, G/xy is planar. Now, define $X = N_G(x) \setminus \{y\}$ and $Y = N_G(y) \setminus \{x\}$. Fix a plane drawing of G/xy. Let F be the face of $G/xy - v_{xy}$ containing v_{xy}. Note that $G/xy - v_{xy}$ is 2-connected (at least). So, f is bounded by a cycle C.

Label the vertices $C \cap X$ as $x_1, x_2, \ldots$. If Y is completely contained in one of these subfaces then we're done. If y is adjacent to two vertices in different subfaces, which do not lie in X, then we can contract to get a $K_{3,3}$ (see book for details), a contradiction. This works so long as Y is adjacent to at least one non-X vertex.

Assume instead that $Y \subseteq X$ and $|X \cap Y|$. By a similar argument (again, see book for details), G contains a K^5, a contradiction. All cases are now covered. □

Remark: To show **Kuratowski's Theorem**, we need to drop the "3-connected" hypothesis in the previous lemma.

Lemma 4.4.4: Let X be a set of 3-connected graphs. Let G be a graph with $\kappa(G) \leq 2$ (i.e., G is at most 2-connected). Let $G_1, G_2 \subseteq G$ satisfy

$$G_1 \cup G_2 = G$$
$$|G_1 \cap G_2| = \kappa(G).$$

(Note that the union is taken over vertices *and* edges.) If G is edge-maximal without a topological minor in X, then so are G_1 and G_2, and $G_1 \cap G_2 = K^2$.

Proof: Set $S := V(G_1 \cap G_2)$. Note that every $v \in S$ has a neighbor in every component of $G_i - S$ for $i = 1, 2$ (otherwise, $S \setminus \{v\}$ would disconnect G, a contradiction to $\kappa(G) = |S|$). Also note that adding any new edge results in a topological minor in X.

If $S = \emptyset$, then G is disconnected. However, we can now add an edge between them (a bridge), which can't be part of a topological minor. Hence G is not edge-maximal, a contradiction.

If $S = \{v\}$, then choose neighbors $v_i \in G_i \setminus \{v\}$, for $i = 1, 2$, which exist by the earlier observation that every $v \in S$ has a neighbor in every component of $G_i \setminus S$ for $i = 1, 2$. Consider $G + v_1v_2$. Suppose we did have a topological minor in X. Then, the branch sets can cross from G_1 to G_2 at most twice (once through v_1v_2 and once through v), but all other branch sets must lie in either G_1 or G_2, since if both G_1 and G_2 had a non-crossing branch set, then we could delete the two branch sets going through v and v_1v_2 to disconnect it, which contradicts the fact that all topological minors in X are 3-connected. So, edge-maximality is contradicted. Hence, $|S| \neq 1$.

Suppose $S = \{x, y\}$. Suppose first that $xy \notin E(G)$. Consider $G + xy$. If $G + xy$ contains a topological minor $x \in X$, then, again, assume all branch sets meet G_1. However, G_2 has an xy path. So, G has a topological minor from X, which is a contradiction. Hence we know that $xy \in E(G)$. It remains to show that G_1 and G_2 are both also edge-maximal. Add a new edge e to G_1. Then, we can pull this minor back into G_1, showing that G_1 is edge-maximal without a topological minor in X. Analogously, so is G_2. □

Lemma 4.4.5: If $|G| \geq 4$ and G is edge maximal without a topological K^5 or $K_{3,3}$ minor, then G is 3-connected.

Proof: (induction on $|G|$) If $|G| = 4$, this is shown easily. Now suppose $|G| > 4$ and that G is edge-maximal without a topological K^5 or $K_{3,3}$ minor, and $\kappa(G) \leq 2$. Choose G_1, G_2 as in **Lemma 4.4.4**. Then, G_1 and G_2 are planar (by induction, and previous lemmas). We know that $xy \in E(G)$, where x and y are as in the previous lemma. Choose drawings of G_i with xy on the outer face. Choose $z_i \in G_i$ also on the outer face. So, $G + z_1 z_2$ has a plane drawing, because if $G + z_1 z_2$ is planar, it can't have a K^5 or $K_{3,3}$ topological minors, contradicting the edge-maximality of G. $\square$

Remark: Combining all of the above lemmas proves **Kuratowski's Theorem**.

4.6 Plane Duality

Definition: To get the dual of a graph, make a vertex for each face, and connect two such vertices through all edges which are adjacent to both faces. Note that this may result in a multigraph with loops.

Definition: Consider a plane multigraph G and another plane multigraph G^* with faces F and F^*, respectively. They are duals if and only if there exist bijections:

$$\begin{cases} F \to V^* := V(G^*) \\ f \mapsto v^*(f) \end{cases},$$

$$\begin{cases} V \to F^* \\ v \mapsto f^*(v) \end{cases},$$

$$\begin{cases} E \to E^* \\ e \mapsto e^*(e) \end{cases},$$

which satisfy conditions:

(1) $v^*(f) \in f$ (i.e., the vertex of a face lies in the face),

(2) $|e^*(e) \cap G| = |\mathring{e^*} \cap \mathring{e}| = |e \cap G^*| = 1$,

(3) $v \in f^*(v)$ (i.e., each old vertex lies in its new face).

For these bijections to exist, both G and G^* must be connected.

Remark: Any two duals of G are topologically equivalent. So, we can say "the dual" instead of "a dual".

Remark: $(G^*)^* = G$.

Proposition 4.6.1: For any connected plane multigraph G, the set $C \subseteq E(G)$ forms a cycle if and only if

$$C^* := \{e^* : e \in C\}$$

is a minimal cutset of G^*.

Proof: Two vertices $v^*(f_1)$ and $v^*(f_2)$ lie in the same component of $G^* - C^*$ if and only if f_1 and f_2 lie in the same region of $\mathbb{R}^2 - C$.

Conversely, let $D \subseteq E(G)$ be such that D^* is a cutset of G^*. If D has no cycle, then it's a forest, and so it has 1 face, which means that D^* is not a cutset, which is a contradiction.

Therefore, D contains a cycle, and thus if D is minimal, then D is precisely the edges of this cycle. $\square$

Remark: So, taking duals interchanges cycles and minimal cutsets:

$$[\text{ cycles }] \overset{*}{\longleftrightarrow} [\text{ minimal cutsets }]$$

Definition: G^* is the abstract dual of G if

$$E(G^*) \overset{*}{\longleftrightarrow} E(G)$$

and the minimal cutsets of G^* correspond to the cycles of G.

Theorem 4.6.3: (Whitney, 1933) A graph is planar if and only if it has an abstract dual.

Proof:

($\Longrightarrow$) If G is planar, then every component of G has a plane dual. Each of the duals has one vertex for the outer face. Pasting the outer face vertices together gives an abstract dual. All edges in a minimal cutset have to come from one component's dual, so it is a cycle. □

($\Longleftarrow$) Proof omitted. Uses algebraic graph theory. □

Chapter 5

Coloring

5.1 Coloring Maps and Planar Graphs

Definition: A coloring of G by the set S is a mapping

$$c : V(G) \to S$$

such that

$$c(v) \neq c(w)$$

if $v \sim w$ in G. If G has an S-coloring, then we say that G is $|S|$-colorable.

Definition: The smallest k so that G is k-colorable is called its chromatic number, and is denoted $\chi(G)$.

Remark: The 2-colorable graphs are exactly the bipartite graphs.

Four Color Theorem: Every planar graph is 4-colorable.

Remark: The above theorem was first stated in 1852. It was first (correctly) proved by Appel-Hakin in 1976. The paper was 400 pages long, and additionally used a computer program to check other cases.

Remark: Why are planar graphs so nice?

- Euler's Formula: $\texttt{vertices} + \texttt{faces} = \texttt{edges} + 2$.
- G is and edge-maximal planar graph if and only if it is a plane triangulation.

Remark: Consider a plane triangulation. By double counting the tuples (f, e) with e on the boundary of f, we see that

$$3f = 2e$$

and so

$$f = \frac{2}{3}e.$$

Applying Euler's Formula,

$$n + \frac{2}{3}e = e + 2$$

and so

$$e = 3n - 6.$$

Therefore, for any plane graph G, we have that

$$\texttt{edges} \leq 3n - 6.$$

We can now prove the following theorem.

Six Color Theorem: Every plane graph is 6-colorable.

Proof: Note that the average degree is

$$d(G) = \frac{2|E|}{n} \leq \frac{2(3n-6)}{n} = 6 - \frac{12}{n} < 6.$$

So, there is some vertex of degree < 6. Remove this vertex. Color the resulting graph by induction. Add the vertex back. It has at most 5 neighbors, so there is a color left over for this vertex. □

Five Color Theorem: ($\approx$ Kempe, circa 1870. Kempe thought this was a proof of the five color theorem. Heawood found an error in 1890 but showed that Kempe's proof did show 5-colorability.) Every plane graph G is 5-colorable.

Proof: (by induction) Proceed by induction on n. Let $v \in G$ be a vertex v with $d(v) \leq 5$, by the same argument as above. By induction, $G - v$ has a 5-coloring. We're done unless $d(v) = 5$ and each neighbor of v has a different color. Let $\{v_i : i \in [5]\}$ be the neighbors of v, labeled in clockwise order. Using topological notions, v is connected to each of these five neighbors by an arc, which to us means a sequence of straight line segments. Call s_i for $i \in [5]$ the first line segment leaving v heading for neighbor v_i. Consider a disc D which intersects *only* the s_i. Without loss of generality, let v_i be colored with color i.

Claim: Every $v_1 - v_3$ path $P \subseteq G - v$ separates v_2 from v_4 in $G - v$. Equivalent, the cycle

$$C := vv_1Pv_3v$$

separates v_2 from v_4 in G. (Most of proofs of this theorem do not formally prove this claim, but we will using topological arguments.) Consider the two regions of $D \setminus (s_1 \cup s_3)$. These two regions are each contained entirely in a face of C. So, $D \cap S_2$ and $D \cap S_4$ lie in different faces of C. Thus, v_2 and v_4 lie in different faces of C, which proves the claim.

Let $H_{1,3}$ be the subgraph of $G - v$ induced by vertices colored 1 or 3. Let C_1 be the component of $H_{1,3}$ containing v_1. If C_1 does not contain v_3, then we can flip all of the colors in C_1 between 1 and 3, which doesn't change v_3, but does change v_1 into being colored 3. Then, we can set the color of v to 3 and we're done.

Otherwise, C_1 does contain v_3, and so C_1 contains a color-alternating $v_1 - v_3$ path $P \subseteq H_{1,3}$. By the earlier claim, this path separates v_2 from v_4 in $G - v$. Consider the similarly defined graph $H_{2,4}$, the subgraph of G_v induced by vertices colored 2 or 4. Note that $P \cap H_{2,4} = \emptyset$. Hence, there is no $v_2 - v_4$ path in $H_{2,4}$. So, if we split $H_{2,4}$ into two or more components, we know that v_2 and v_4 are in different components. Now, flip all of the colors in the component containing v_2, so that v_2 is colored 4, and make v have color 2. □

5.2 Coloring Vertices

Question: How can we relate the number of edges in a graph G with its chromatic number $\chi(G)$?

Definition: Given a coloring of G, the set of vertices colored by a particular color is called a color class.

Remark: Every color class forms an independent set.

Remark: If $\chi(G) = k$, then in any k-coloring of G, there is an edge between every pair of color classes. Hence, we get the following bound: The number m of edges of G satisfies

$$|E(G)| =: m \geq \binom{k}{2}.$$

Hence,

$$\begin{aligned} m &\geq \frac{1}{2}k(k-1) \\ m &\geq \frac{1}{2}k^2 - \frac{1}{2}k \\ 0 &\geq \frac{1}{2}k^2 - \frac{1}{2}k - m \\ k &\leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2m}. \end{aligned}$$

This proves the following proposition.

Proposition 5.2.1: Every graph G with m edges satisfies

$$\chi(G) \leq \frac{1}{2} + \sqrt{\frac{1}{2} + 2m}.$$

Remark: Recall that the girth of a graph is the length of the smallest cycle. It may seem reasonable that a graph has a small girth if and only if it has a large $\chi(G)$. However, Erdős showed that there exist graphs with arbitrarily large girth and arbitrary large $\chi(G)$. We will prove this in Chapter 12.

Proposition: We have the bound

$$\chi(G) \leq \Delta(G) + 1.$$

This is found by use of the **Greedy Coloring Algorithm:**

```
List the vertices v_1, v_2, ..., v_n.
Give v_i the smallest color not taken by its neighbors in v_1, ..., v_{i-1}.
```

Remark: We have equality in the bound above when G is a complete graph or an odd cycle (or unions of such). We will soon prove that these are the only two such cases.

Remark: In the **Greedy Coloring Algorithm**, we want to list the vertices from high degree to low degree. To do this, choose v_n such that $d(v_n) = \delta(G)$, then remove v_n and repeat. The coloring number of G_n, denoted col(G), is the least number k such that there is a vertex enumeration $v_1, \ldots, v_n$ such that

$$d_{G[v_1,\ldots,v_i]}(v_i) < k$$

for all i. We've already shown the following proposition.

Proposition 5.2.2:

$$\chi(G) \leq \operatorname{col}(G) = \max\{\delta(H) \mid H \leq G\} + 1.$$

Corollary 5.2.3: Every graph G is a subgraph of minimum degree at least $\chi(G) - 1$.

Theorem 5.2.4: (Brooks, 1941) Let G be a connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

> **Proof:** (by induction) Proceed by induction on $|G|$. The base case $|G| = 1$ is trivial. The case $\Delta(G) \leq 2$ is also trivial: these are paths and cycles, and we're specifically excluding odd cycles.
>
> Suppose $\Delta(G) \geq 3$, and that $\chi(G) > \Delta(G)$. We want to show that G is a complete graph.
>
> Take any vertex $v \in G$ and define $H := G - v$. We claim that $\chi(H) \leq \Delta(G)$: every component H' of H satisfies $\chi(H') \leq \Delta(H')$ by induction, unless H' is one of the exceptions. If H' is an exception, then $H' = K^n$ or $H' = C^{2n+1}$. So, v must have been adjacent to some w of H' which then has degree at least $\chi(H')$ in G. This proves the claim.
>
> Since $\chi(H) \leq \Delta(G)$ but $\chi(G) > \Delta(G)$, the neighbors of v use all $\Delta(G)$ colors. Therefore, $d(v) = \Delta(G)$, i.e. v has maximum degree.
>
> Given a Δ-coloring of H and colors i and j, let $H_{i,j}$ denote the subgraph spanned by vertices of colors i and j. Let v_i denote the neighbor of v which has color i. For all $i \neq j$, v_i and v_j lie on a common component $C_{i,j}$ of $H_{i,j}$ (to see this, we use a similar idea as the Kempe chain trick of the 5 Connected Theorem: otherwise, we could swap colors i and j in one of these components and be finished, which would be a contradiction.)

5.3 Coloring Edges

Definition: Define $\chi'(G)$ to be the edge chromatic number of G. This is the smallest number of colors needed such that any two incident edges have different colors.

Remark: We have the following trivial lower bound:

$$\chi'(G) \geq \Delta(G).$$

König's Theorem: (1916) For bipartite graphs G, we have equality in the above bound, i.e, $\chi'(G) = \Delta(G)$.

> **Proof:** (by induction) We use induction on $|E(G)|$. Set $\Delta := \Delta(G)$ and take an edge $xy \in E(G)$. Consider a Δ-edge-coloring of $G - xy$. In our coloring of $G - xy$, there are colors $\alpha, \beta \in \{1, \ldots, \Delta\}$ which are missing at x and y, respectively.
>
> If $\alpha = \beta$, use this color for xy. Otherwise, take the longest walk which starts at x and alternates between α and β edges. We claim that this walk is actually a path, i.e., it can't revisit a vertex: if it did revisit a vertex, this would violate the edge-coloring. Next, we claim that this walk/path does not end at y: otherwise there would be an odd cycle which violates the fact that G is bipartite. So, we can switch β and α along this path, which forces β to be missing at both x and y, allowing us to color xy with color β. $\square$

Vizing's Theorem: (1964) For every graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

Graphs for which $\chi'(G) = \Delta(G)$ are called "class 1 graphs" and graphs for which $\chi'(G) = \Delta(G)+1$ are called "class 2 graphs".

Proof: (by induction) Use induction on $|E(G)|$. Set $\Delta := \Delta(G)$. For the rest of the proof, "coloring" means "$(\Delta+1)$-edge-coloring".

For every $e \in G$, the graph $G - e$ has a coloring (by induction). Every $v \in V(G)$ is missing a color $\beta \in \{1, \ldots, \Delta+1\}$. For any other color α, there is a maximal α/β walk starting at v.

Suppose G has no coloring. Then, for every $xy \in E(G)$, and any coloring of $G - xy$, in which α is missing at x and β is missing at y, the α/β path from y ends at x. (Here we used the same two claims as in the proof of the previous theorem.)

Let $xy_0 \in E(G)$ be an edge. Let α be missing at x in some coloring of $G - xy_0$ and denote this coloring by

$$c_0 : E(G - xy_0) \to \{1, \ldots, \Delta+1\}.$$

Let $y_0, y_1, \ldots, y_k$ be a maximal sequence of distinct neighbors of x in G so that $c_0(xy_i)$ is missing at y_{i-1}.

For every $G - xy_i$, we define a coloring by

$$c_i(e) := \begin{cases} c_0(xy_{j+1}), & e = xy_j,\ j \in \{0, 1, \ldots, i-1\} \\ c_0(e), & \text{otherwise} \end{cases}$$

The map c_i deletes xy_i and shifts all colors before it down.

Let β be missing at y_k in c_0. In c_k, β is still missing at y_k. So, x has a β edge in every c_i. So, x has a β edge in every c_i. So, some edge xy_i is colored β in c_0. Take an α/β path P from y_k in $G - xy_k$ colored by c_k. Then, P ends at x in a β edges, and

$$\beta = c_0(xy_i) = c_k(xy_{i-1}).$$

So, y_{i-1} is missing β in c_0.

Look at the α/β path from y_{i-1} in $G - xy_{i-1}$ colored by c_{i-1}. This path starts with $y_{i-1}Py_k$. But in c_0 and c_{i-1} there is no β edge at y_k. $\square$

5.4 List Coloring

Definition: Let $G = (V, E)$ be a graph. Consider a family of sets $(S_v)_{v \in V}$. We say that G is (S_v)-colorable if there is a valid coloring c such that $c(v) \in S_v$ for all v. We say that G is k-list-colorable if G is (S_v)-colorable for all families $(S_v)_{v \in V}$ with $|S_v| = k$ for all $v \in V$.

Definition: The least k for which G is k-list colorable is its list-chromatic (or choice) number, $\text{char}(G)$.

Remark: This makes coloring harder: $\text{char}(G) \geq \chi(G)$, for all graphs G.

Definition: The analogous notion is edge-list-coloring, which has the same property $\text{char}'(G) \geq \chi'(G)$.

Remark: Brooks' Theorem holds for list-colorings:

$$\text{char}'(G) = \Delta(G) \text{ or } \Delta(G) + 1.$$

Remark: The following result still holds:

$$[\text{large char}(G)] \Longrightarrow [\text{there is a subgraph with large average degree}].$$

Theorem: (Alon, 1993) There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $d(G) \geq f(k)$, then $\text{char}(G) \geq k$. For $\chi(G)$, this is false: consider $K_{n,n}$.

Theorem: (Thomassen, 1994) Every planar graph is 5-list-colorable.

Proof: We use induction on $|G| \geq 3$ to prove a stronger statement:

> $(*)$: Suppose that every inner face of G is bounded by a triangle, and its outer face by the cycle $C = v_1 \cdots v_k v_1$. Suppose further that v_1 is colored 1 and v_2 is colored 2. Suppose that every other vertex of C has a list of 3 colors, and each vertex of $G - C$ has a list of 5 colors. Then, this coloring can be extended from these lists.

From $(*)$, we get the theorem: Extend G to a plane triangulation and suppose the outer face is bounded by $v_1v_2v_3v_1$. Color v_1, color v_2 different than v_1, and use $(*)$ to extend the coloring.

In the base case, $|G| = 3$, and so G is a triangle. By inspection $(*)$ holds. Now assume $|G| \geq 4$.

(**Case 1:**) Assume C has a chord vw. The edge vw splits G into two graphs G_1 and G_2 (where the chord is in both graphs). Color G_1 by induction. Then color G_2 by induction.

(**Case 2:**) Assume C has no chord. Label the neighbors of v_k as $v_1, u_1, \ldots, u_m, v_{k-1}$. It follows that (see book for details) that these neighbors lie on a path (with possible relabeling of the u_i) $P = v_1 u_1 u_2 \cdots u_m v_{k-1}$. Set the cycle $C' = P \cup (C - v_k)$. v_k has 3 colors. At least two of these aren't the color 1. Delete these two colors from each u_i, so that each u_i has a list of 3 colors. Delete v_k and use induction with the new outer cycle C'. Put v_k back in. It has two possible colors, which don't conflict with v_1 or any u_i. The only other neighbor is v_{k-1}, so we can pick a color v_k.

This completes the proof. □

List Coloring Conjecture: $\text{char}'(G) = \chi'(G)$ for all graphs G.

Remark: This is another example of how edge coloring behaves very differently than vertex coloring.

Definition: Let G be a graph and let D be an <u>orientation</u> of G. So, an edges in G becomes an <u>oriented edge</u> in D. We say that D is an <u>oriented graph</u>. We have the following notation:

$$N^+(v) := \{u \mid D \text{ has an edge } v \to u\},$$
$$d^+(v) := \left|N^+(v)\right|.$$

Definition: We say that $U \subseteq D$ is a <u>kernel</u> if

(i) U is an independent set,

(ii) for every $v \in D - U$, there is an edge from v to a vertex in U.

If $D \neq \emptyset$, then its kernel (if it has one) is also nonempty.

Lemma 5.4.3: Let H be a graph and let $(S_v)_{v\in H}$ be a family of lists. If H has an orientation D with $d^+(v) < |S_v|$ for every v, and every induced subgraph has a kernel, then H can be colored with the lists $(S_v)_{v\in H}$.

> **Proof:** We use induction on $|H|$. The base case $|H| = 0$ is trivial. Consider a graph H with $|H| > 0$. Let α be a color occurring in one of the lists. Let D be an orientation as specified by the hypothesis.
>
> Define D' as the graph induced by vertices v with $\alpha \in S_v$. Let U be the kernel of D'. Color every vertex in U by α, and remove α from the other lists of vertices in D' to get lists $(S'_v)_{v\in V}$. Note that $d^+(v) < |S'_v|$ for $v \in D - U$. So, by induction, we can color $H - U$ with the lists $(S'_v)_{v\in V}$. □

Galvin's Theorem: (1995) If G is a bipartite graph, then $\text{char}'(G) = \chi'(G) = \Delta(G)$.

> **Proof:** Let G have bipartition $X \cup Y$. Let $\chi'(G) = k$. Let $c : E(G) \to [k]$ be a k-edge coloring of G. We know that $\text{char}'(G) \geq \chi'(G)$. Our goal is to use **Lemma 5.4.3** to show that the line graph of G (denoted H) is k-choosable.
>
> We assign the following orientation to the edges of H. Suppose $e, e' \in E(G)$ meet, and that $c(e) < c(e')$. Then, if e and e' meet in X, we assign $e' \to e$. If e and e' meet in Y, we assign $e \to e'$.
>
> So, what is $d^+_D(e)$? If $c(e) = i$, then $e \to e'$ for edges e' that meet e in X with $c(e') \in [i-1]$, and for edges e' that meet e in Y with $c(e') \in [k] \setminus [i]$. So, $d^+_D(e) < k = |S_v|$. This satisfies half of the **Lemma**.
>
> How about the kernel condition? We claim that every $D' \subseteq D$ has a kernel. Proceed by induction on $|D'|$. If $|D'| = 0$ or $|D'| = 1$, then it's true. Now set $E' = V(D') \subseteq E(G)$. For $x \in X$ at which E' has an edge, let $e_x \in E'$ be the edge at X with minimum c-value. Let $U = \{e_x \mid x \in X\}$. Every edge $e' \in E' \setminus U$ meets some edge in U, and each of these edges $e'e$ is oriented $e' \to e$. If U is an independent set, it is a kernel and we're done.
>
> Let $e, e' \in U$ be adjacent. Then, e and e' meet in Y. Suppose $c(e) < c(e')$. So $e \to e'$. By induction $D' - e$ has a kernel U'. If $e' \in U'$, we're done. If not, then U' has an edge e'' with $e' \to e''$. We have two situations.
>
> If e' and e'' meet in X, then $c(e'') < c(e')$. Then, e' isn't the minimal c-value for its vertex in x, a contradiction to the definition of U.
>
> If e' and e'' meet in Y, then $c(e') < c(e'')$. Then, $c(e) < c(e') < c(e'')$ and so $c(e) < c(e'')$, so that $e \to e''$ and U is a kernel of D'. □

5.5 Perfect Graphs

Remark: All subgraphs in this section are induced subgraphs, unless otherwise stated.

Definition: $\underline{\omega(G)}$ is the greatest r such that $K^r \subseteq G$. This is called the clique number.

Definition: $\underline{\alpha(G)}$ is the greatest r such that $\overline{K}^r \subseteq G$. This is called the independence number.

Definition: G is perfect if $\chi(H) = \omega(H)$ for all induced $H \subseteq G$.

Remark: The class of perfect graphs is closed under induced subgraphs, but is not closed under minors.

Example: The cycle of length 6 is perfect, but the cycle of length 5 (which is its minor) is not.

Definition: A graph is called chordal if all of its cycles of length at least 4 have a chord. These graphs are sometimes called triangulated graphs.

Remark: Suppose G has induced subgraphs G_1, G_2, and S. If $G = G_1 \cup G_2$ and $S = G_1\ capG_2$, then G is formed by "pasting together" G_1 and G_2 along S.

Proposition 5.5.1: A graph is chordal if and only if it can be constructed recursively by pasting together along complete graphs, starting from complete graphs.

Proof:

($\Longleftarrow$) Consider a graph G constructed by pasting G_1 and G_2 along the complete graph $S = G_1 \cap G_2$. Let C be an induced cycle of G of length at least 4. Then, C lies entirely in G_1 or G_2: otherwise, it has two non-adjacent vertices in S which would have to be connected. This is a contradiction since G_1 and G_2 are chordal by induction. □

($\Longrightarrow$) Proceed by induction on $|G|$. The base case is trivial. Let G be a chordal graph. If G is complete, we're done immediately. So, assume that a and b are non-adjacent vertices of G. Let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal set of vertices separating a from b (of course, X may be empty, in which case we're pasting along a K^0, and we're done by induction).

By construction $G - X$ is disconnected. Let C be the component of $G - X$ containing a. Define

$$\begin{aligned} G_1 &:= G[V(C) \cup X], \\ G_2 &:= G - C, \\ S &:= G[X]. \end{aligned}$$

We need to show that S is complete. Suppose to the contrary that there are nonadjacent $s, t \in S$ such that s and t are non-adjacent. Let P_1 be a minimal $s-t$ path in G_1 and let P_2 be a minimal $s - t$ path in G_2. But, then we've made an induced cycle of length at least 4, a contradiction. □

Proposition 5.5.2: Chordal graphs are perfect.

Proof: We proceed by induction on the recursive construction in **Proposition 5.5.1**. In the base case, the graphs are complete, hence obviously chordal.

Now suppose that G is constructed by pasting G_1 and G_2 along the complete graph S. Let H be an induced subgraph of G. Define

$$\begin{aligned} H_1 &:= H \cap G_1, \\ H_2 &:= H \cap G_2, \\ T &:= H \cap S. \end{aligned}$$

T is still a complete graph. First, $\omega(H) = \max\{\omega(H_1), \omega(H_2)\} \geq |T|$. By induction, color H_1 with $\omega(H_1)$ colors and color H_2 with $\omega(H_2)$ colors. We can permute the colors so that they agree on T, and then we're done. □

Weak Perfect Graph Theorem: (Lovász, 1972) G is perfect if and only if $\overline{G}$ is perfect.

Theorem: (Lovász, 1972) G is perfect if and only if $|H| \leq \alpha(H)\omega(H)$ for all induced subgraphs H of G.

Strong Perfect Graph Theorem: (Chutnovsky, Robertson, Seymour, Thomas, 2002) G is perfect if and only if G has no induced C^5, C^7, C^9, $\overline{C^5}$, $\overline{C^7}$, or $\overline{C^9}$.

Chapter 7

Extremal Graph Theory

7.1 Substructures in Dense Graphs

Remark: In this chapter, we explore how we can relate global invariants of graphs with local structures which exist in the graphs. For example, how many edges must a graph have to ensure the existence of a K^r subgraph? This kind of question falls under the category of extremal graph theory.

Example: If we are trying to ensure an H minor, the answer is that we need $|E| \geq c|V|$ for some c. So, we just need to push $d(G) = \frac{2|E|}{|V|}$ up high enough?

Example: What if we want an H subgraph? Consider $H = C^4$. Erdős constructed graphs with arbitrary large chromatic number and girth. Additionally, every graph G has a subgraph with $\delta(G) \geq \chi(G) - 1$. In Chapter 1, we showed that large minimum degree implies high connectivity. These all show that the typical global invariants being large will not force the existence of a C^4 subgraph. The same reasoning holds for any H containing any cycle of length at least 4.

Remark: If we let our parameters depend on $n = |G|$, then the above is possible. For example, if we require $\delta(G) \geq n - 1$, then $G = K^n$ and certainly G contains a C^4 subgraph if $n \geq 4$. We need to insist on positive edge density:

$$\frac{|E|}{\binom{|V|}{2}} > 0.$$

Of course, for any particular graph, this number is strictly positive, but when we refer to dense graphs, we're really talking about a sequence of graphs:

$$\inf\left(\text{edgedensity}(G_i)\right) > 0.$$

When we refer to sparse graphs, we mean

$$\sup\left(\text{edgedensity}(G_i)\right) = 0.$$

This terminology is frequently abused.

Definition: Let H be a fixed graph and $n \geq |H|$. What is the greatest number of edges an n-vertex graph can have without containing an H subgraph. A graph with this many edges any no copy of H is called extremal for H. This number of edges is denoted $\text{ex}(n, H)$.

Remark: We showed in the above remark that if H is not a forest, then $\text{ex}(n, H)$ does not grow linearly with respect to n.

Definition: Say that we want to find the extremal graphs which do not have a $H = K^r$ subgraph? If $r = 3$, we can split G into a complete bipartite graph on sets of equal (or off by 1) size. Generalizing for r, we can split G into an $(r-1)$-partite graph such that all parts have equal (or off by 1) size. This is called the Turan graph $T^{r-1}(n)$.

Definition: Define $t^{r-1}(n) = \|T^{r-1}(n)\|$.

Turan's Theorem: (1941) For all $r > 1$ and n, every graph $G \not\supseteq K^r$ with n vertices and $t^{r-1}(n)$ edges is a $T^{r-1}(n)$.

Proof: We proceed by induction on n. If $n \leq r-1$, then by definition $T^{r-1}(n)$ is the complete graph on $< r$ vertices, so we're done.

Suppose $n \geq r$. Take G to be extremal for K^r. Then G is edge-maximal, and so $G \supseteq K^{r-1}$. Call this subgraph K. Well,

$$\|G - K\| \leq t^{r-1}(n - r + 1),$$

since every vertex of $G - K$ has at most $r-2$ neighbors in K. Now,

$$\|G\| \leq t^{r-1}(n - r + 1) + (n - r + 1)(r - 1) + \binom{r-1}{}.$$

The Turan graph has exactly this many edges, thus

$$|G| = t^{r-1}(n).$$

So, $T^{r-1}(n)$ is extremal for K^r. Since G is extremal, it has $t^{r-1}(n)$ edges.

Let

$$V(K) = \{x_1, \ldots, x_{r-1}\}.$$

For $i = 1, \ldots, r-1$, define

$$V_i = \{v \in G : vx_i \notin E.$$

Note that $x_i \in V_i$. If some V_i contained an edge, then those two vertices plus $\{x_1, \ldots, x_{r-1}\} \setminus \{v_i\}$ would form a K^r. So, G is an $(r-1)$-partite graph, and by algebra its parts differ by 1. □

Exercise: $t_{r-1}(n) \leq \frac{1}{2}n^2 \cdot \frac{r-2}{r-2}$.

Erdős-Stone Theorem: (1946) For all integers $r \geq 2$ and $s \geq 1$ and every $\epsilon > 0$, there is an integer n_0 such that every graph on $n \geq n_0$ vertices with at least $t_{r-1}(n) + \epsilon n^2$ edges contains a K_s^r (as a subgraph), i.e., a complete r-partite graph with s vertices in each part.

Corollary: When we combine **Turan's Theorem** and the above exercise, it follows that

$$\frac{\text{ex}(n, K^r)}{\binom{n}{2}} \to \frac{r-2}{r-1}.$$

What about other avoiding graphs?

Corollary 7.1.3: For every graph H with at least one edge,

$$\frac{\text{ex}(n,H)}{\binom{n}{2}} \to \frac{\chi(H)-2}{\chi(H)-1}.$$

Proof: Let $r = \chi(H)$. Then, H can't be colored with $r-1$ colors. So, $H \not\subseteq T^{r-1}(n)$. Thus,

$$\text{ex}(n,H) \geq t_{r-1}(n)$$

and so

$$\lim_{n\to\infty} \frac{\text{ex}(n,H)}{\binom{n}{2}} \geq \frac{\chi(H)-2}{\chi(H)-1}.$$

Color H with r colors and let s be the size of the largest color class. Then, H is contained (as a subgraph) K_s^r. By Erős-Stone, we can fix $\epsilon > 0$ and let $n \geq n_0$ to show that if we have at least $t_{r-1}(n) + \epsilon n^2$ edges, then we have a K_s^r and hence an H subgraph.

So,

$$\frac{t_{r-1}(n)}{\binom{n}{2}} \leq \frac{\text{ex}(n,H)}{\binom{n}{2}} \leq \frac{\text{ex}(n,K_s^r)}{\binom{n}{2}} \leq \frac{t_{r-1}(n)+\epsilon n^2}{\binom{n}{2}} = \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{2\epsilon}{1-\frac{1}{n}} \leq \frac{t_{r-1}(n)}{\binom{n}{2}} + 4\epsilon.$$

Thus, letting $n \to \infty$ and then $\epsilon \to 0$, the upper and lower bounds match. □

Remark: For $H = K_{s,s}$,

$$c_1 n^{\left(2-\frac{2}{r+1}\right)} \leq \text{ex}(n,K_{s,s}) \leq c_2 n^{\left(2-\frac{1}{r}\right)}.$$

If H is a forest, then

$$\text{ex}(n,F) \leq cn.$$

Conjecture (Erdős - Sós, 1963),

$$\text{ex}(n,\text{Tree}) \leq \frac{1}{2}(\text{edges in tree} - 1)n.$$

Definition: Let $X, Y \subseteq V(G)$ be disjoint. Define $\|X,Y\|$ to be the number of $X-Y$ edges. Define the density by

$$d(X,Y) := \frac{\|X,Y\|}{|X||Y|}.$$

Definition: Fix some $\epsilon > 0$. The pair (A,B) of disjoint subsets of $V(G)$ is ϵ-regular if for all $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have

$$|d(X,Y) - d(A,B)| < \epsilon.$$

Remark: In an ϵ-regular pair, the edges are distributed fairly uniformly.

7.4 Szemerédi's Regularity Lemma

Definition: Suppose $\{V_0, V_1, \ldots, V_k\}$ is a partition of $V(G)$. Call V_0 the "exceptional part". This is an ϵ-regular partition of G if:

(1) $|V_0| \le \epsilon |V(G)|$,

(2) $|V_1| = \cdots = |V_k|$,

(3) all but at most ϵk^2 of the pairs (v_i, v_j) are ϵ-regular.

Szemerédi's Regularity Lemma: For every $\epsilon > 0$ and integer $m \ge 1$, there is an integer M such that every graph of order at least m admits and ϵ-regular partition $\{V_0, V_1, \cdots, V_k\}$ with $m \le k \le M$.

Proof: See book.

Lemma 7.4.1: Let (A, B) be an ϵ-regular pair of density d. Take $Y \subseteq B$ with $|Y| \ge \epsilon |B|$. Then, all but at most $\epsilon |A|$ of the vertices in A have (each) at least $(d - \epsilon)|Y|$ neighbors Y.

Proof: Let $X \subseteq A$ be the set of vertices in A with fewer than $(d - \epsilon)|Y|$ neighbors in Y. So,

$$d(X, Y) < \frac{|X|(d - \epsilon)|Y|}{|X||Y|} = d - \epsilon.$$

Therefore, $|X| < \epsilon |A|$ because (A, B) is ϵ-regular. □

Definition: Let G have ϵ-partition $\{V_0, V_1, \ldots, V_k\}$. Suppose $|V_1| = \cdots = |V_k| =: \ell$. Given $d \in (0, 1]$, let R be the graph on $V_1, \ldots, V_k$ in which two parts are adjacent if they are an ϵ-regular pair of density at least d. We say that R is a regularity graph with parameters ϵ, ℓ, and d.

Definition: Given $s \in \mathbb{N}$ replace V_i in R by an independent set V_i^s of s vertices and replace the edges of R by complete bipartite graphs between these s-sets. Denote the graph we obtain by R_s.

Lemma 7.4.2: For all $d \in (0, 1]$ and $\Delta \ge 1$, there exists an $\epsilon_0 > 0$ with the following property: If G is a graph and $\Delta(H) \le \Delta$, $s \in \mathbb{N}$, and R is a regularity graph of G with parameters $\epsilon \le \epsilon_0$ and $\ell \ge s/\epsilon_0$, and d, then

$$[H \subseteq R_s] \implies [H \subseteq G].$$

Proof: Given d and Δ, choose $\epsilon_0 < d$ small enough so that

$$\frac{\Delta + 1}{(d - \epsilon_0)^\Delta} \le 1.$$

Let G, H, s, and R be as states. Let $\{v_0, v_1, \ldots, v_k\}$ be the ϵ-regular partition of G that gives rise to R. So,

$$|V_1| = \cdots = |V_k| = \ell \ge \frac{s}{\epsilon_0}.$$

Suppose H is a subgraph of R_s with vertices $u_1, \ldots, u_k$. Each u_i lies in one of the s-sets V_j^s of R_s. This defines a map $\sigma : i \mapsto j$. We want an embedding

$$u_i \mapsto v_i \in V_{\sigma(i)} \subseteq G.$$

Thus, we need the v_i to be distinct and $v_i \sim v_j$ whenever $u_i \sim u_j$. We choose the v_i inductively. Throughout the induction, maintain a target set $Y_i \subseteq V_{\sigma(i)}$ for each i. Initially, $Y_i = V_{\sigma(i)}$. Once v_i is chosen, $Y_i = \{v_i\}$. Whenever we choose a v_j with $j < i$, if $u_j \sim u_i$, delete all vertices of v_i that are not adjacent to v_j. The evolution of Y_i is

$$V_{\sigma(i)} = Y_i^0 \supseteq \cdots \supseteq Y_i^i = \{v_i\}.$$

We need to ensure that $|Y_i^{i-1}| > 0$. Suppose we are choosing v_j, and consider each $i > j$ with $u_j \sim u_i$. There are at most Δ such indices i. For each of these i, we want to make

$$Y_i^j = N(v_j) \cap Y_i^{j-1}$$

large.

Recall **Lemma 7.4.1** above. Applying this with $A = V_{\sigma(j)}$, $B = V_{\sigma(i)}$ and $Y = Y_i^{j-1}$, we are done unless $|Y_i^{j-1}| \leq \epsilon |V_{\sigma(j)}| = \epsilon\ell$. Otherwise the lemma says all but at most $\epsilon\ell$ choices of v_j will be such that

$$|Y_i^j| = |N(v_j) \cap Y_i^{j-1}| \geq (d-\epsilon)|Y_i^{j-1}|. \qquad (*)$$

Do this simultaneously for all of the at most Δ choices of i for which $u_i \sim u_j$. This means that we must avoid $\leq \Delta\epsilon\ell$ of v_j from $V_{\sigma(j)}$, and thus also from $Y_j^{j-1} \subseteq V_{\sigma(j)}$. The goal is to show

$$|Y_j^{j-1} - \Delta\epsilon\ell \geq s.$$

If we can show this, then a good choice of v_j exists: since $\sigma(j') = \sigma(j)$ for at most $s-1$ vertices u'_j with $j' < j$, so there will be an unused v_j left for u_i. Now,

$$\begin{aligned} |Y_i^j| - \Delta\epsilon\ell &\geq (d-\epsilon)^\Delta \ell - \Delta\epsilon\ell \\ &\geq (d-\epsilon_0)^\Delta \ell - \Delta\epsilon_0\ell \\ &= ((d-\epsilon_0)^\Delta - \Delta\epsilon_0)\ell \\ &\geq \epsilon_0 \ell \\ &\geq s. \end{aligned}$$

Thus we can avoid bad v_j forever. □

Proof of Erdős-Stone: Let $|G| = n$ and $\|G\| \geq t_{r-1}(n) + \gamma n^2$. Set $d := \gamma$ and $\Delta := \Delta(K_s^r) = (r-1)s$. We may assume $\epsilon_0 < \frac{\gamma}{2} < 1$.

Let $m > \frac{1}{\gamma}$ and choose $\epsilon \leq \epsilon_0$ small enough so that

$$\delta := 2\gamma - \epsilon^2 - 4\epsilon - d - \frac{1}{m} > 0.$$

Suppose that

$$n \geq \frac{Ms}{\epsilon_0(1-\epsilon_0)} \geq M \geq m.$$

The **Regularity Lemma** gives a family $\{V_0, V_1, \ldots, V_k\}$ with $m \leq k \leq M$. Set

$$\ell := |V_1| = \cdots = |V_k|.$$

Then, $n \geq k\ell$ and

$$\ell = \frac{n - |v_0|}{k} \geq \frac{n - \epsilon n}{k} \geq \frac{n - \epsilon n}{M} = n\frac{(1-\epsilon)}{M} \geq \frac{Ms}{\epsilon_0(1-\epsilon_0)} \frac{(1-\epsilon}{M} \geq \frac{s}{\epsilon_0}.$$

Let R be the regularity graph with parameters ϵ, ℓ, d. The lemma applies to R, so it suffices to show that R has a K^r. The goal is to show that $\|R\| \geq t_{r-1}(k)$. We know that

$$t_{r-1}(k) = \frac{k^2}{2}\frac{r-2}{r-1}.$$

How many edges involve V_0? Inside V_0, there are at most

$$\binom{|V_0|}{2} \leq \frac{1}{2}\epsilon^2 n^2$$

edges. Outside of V_0, there are at most

$$|V_0 k\ell| \leq \epsilon n k \ell$$

edges.

There are also regular pairs between non-regular pairs (V_i, V_j). By the **Regularity Lemma**, there are at most ϵk^2 such pairs, each with at most ℓ^2 edges, for a total of at most $\epsilon k^2 \ell^2$ edges.

There are also edges between pairs (V_i, V_j) which are ϵ-regular but have insufficient density to be part of the regularity graph. Each such pair has strictly fewer than $d\ell^2$ edges, and there are at most $\frac{1}{2}k^2$ such pairs, for a total of strictly fewer than $\frac{1}{2}k^2 d\ell^2$ edges.

There are also edges inside of a particular V_i for $i \neq 0$. There are at most $\frac{1}{2}\ell^2 k$ such edges.

The remaining edges are between pairs (V_i, V_j) of sufficient density. The only upper bound for this number of edges is $\|R\|\ell^2$, where $\|R\|$ is the number of edges in the regularity graph.

Therefore,

$$\|G\| \leq \frac{1}{2}\epsilon^2 n^2 + \epsilon n k \ell + \epsilon k^2 \ell^2 + \frac{1}{2}k^2 d\ell^2 + \frac{1}{2}\ell^2 k + \|R\|\ell^2.$$

Hence,

$$\begin{aligned}
\|R\| &\geq \frac{\|G\| - \frac{1}{2}\epsilon^2 n^2 - \epsilon n k \ell - \epsilon k^2 \ell^2 - \frac{1}{2}k^2 d\ell^2 - \frac{1}{2}\ell^2 k}{\ell^2} \\
&= \frac{k^2}{2}\left(\frac{\|G\| - \frac{1}{2}\epsilon^2 n^2 - \epsilon n k \ell - \epsilon k^2 \ell^2 - \frac{1}{2}k^2 d\ell^2 - \frac{1}{2}\ell^2 k}{\frac{k^2}{2}\ell^2}\right) \\
&\geq \frac{k^2}{2}\left(\frac{t_{r-1}(n) + \gamma n^2 - \frac{1}{2}\epsilon^2 n^2 - \epsilon n k \ell}{\frac{n^2}{2} - 2\epsilon - d - \frac{1}{k}}\right) \\
&\geq \frac{k^2}{2}\left(\underbrace{\frac{t_{r-1}(n)}{\frac{n^2}{2}}}_{=\frac{r-2}{r-1}} + \underbrace{2\gamma - \epsilon^2 - 4\epsilon - d - \frac{1}{m}}_{=\delta>0}\right) \\
&\geq \frac{k^2}{2}\left(\frac{r-2}{r-1} + \delta\right) \geq t_{r-1}(k). \quad \square
\end{aligned}$$

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