Monday, Dec 5 - Fall '22
Lecture #39

Announcements / Reminders

* Wiley Plus #14 due Wed. night
* Quiz 12 in discussion on Thursday

(4.7, 5.1)

* Course Evaluations are open. If 90% of the class does them, I will give everyone two bonus points.

until Sunday

* Final Exam, 1pm-3pm on Monday, in this room

Note:

$f(a) = 0$

$g(a) = 0$

so

$$\frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} \Rightarrow$$

$$\frac{0 + f'(a)(x-a)}{0 + g'(a)(x-a)} \Rightarrow$$

$$\frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} \Rightarrow$$

$$\frac{f'(a)}{g'(a)}$$

If $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$, then we can use $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ to evaluate our limit.

If $g'(a) = 0$, then we can take another derivative of both $f'(x)$ and $g'(x)$.

Sometimes, we need to do some algebra, specifically to get our limit to look like something we use L'Hopitals Rule on:

Ex 8 $\lim_{x\to\infty} xe^{-x} = \infty \cdot e^{-\infty} = \infty \cdot 0$

$$= \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

Apply L'Hopital's Rule! $= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

Back in Chapter 1, we saw that some functions eventually are always larger than other functions. For example, eventually all exponential functions, with a base greater than 1 will grow faster, and larger, than polynomials.

e^x and x^2 and $e^x > x^2$ for large x.

We would say e^x dominates over x^2 as $x \to \infty$.

Zoom in really close on the above graph near $x=0$, we see we have 2 straight(ish) lines.

Local Linearization:

$$\underset{(\text{near } a)}{f(x)} \approx f(a) + f'(a)(x-a)$$

Note: $f'(x) = 2e^{2x}$

$f'(0) = 2$

$\boxed{\text{From Ex 4:}}$ $\underset{\text{near } a=0}{f(x)} \approx 0 + 2(x-0) = 2x$

$\underset{\text{near } a=0}{g(x)} \approx x$

$$\lim_{x \to 0} \frac{2x}{x} = 2$$

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ then we can substitute in the local linearizations to evaluate the limits

L'Hopital's Rule:

If f and g are differentiable functions, and $f(a)=g(a)=0$, and $g'(a)\neq 0$ then:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

L'Hopital's Rule and the Quotient Rule are two seperate things.

We only use L'Hopital's Rule for evaluating limits.

Ex 5: $\lim_{x\to 0}\frac{\sin(x)}{x}=\frac{\sin(0)}{0}=\frac{0}{0}$

$$=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin(x)]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{\cos(x)}{1}=\frac{1}{1}=1$$

Ex 4 Part 2, the L'Hopital Bugaloo

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}=\frac{0}{0};\ \lim_{x\to 0}\frac{2e^{2x}}{1}=\frac{2e^{0}}{1}=2$$

$\boxed{\text{Ex 6}}$

$$\lim_{t\to 0} \frac{e^t - t - 1}{t^2} = \frac{e^0 - 0 - 1}{0} = \frac{0}{0}$$

Use L'Hopitals! $\lim_{t\to 0} \frac{e^t - 1}{2t} = \frac{0}{0} \xrightarrow{\text{L'Hopital}}$

Use L'Hopitals (again)! $\lim_{t\to 0} \frac{e^t}{2} = \boxed{\frac{1}{2}}$

L'Hopital's Rule applies in other places besides $\frac{0}{0}$

Also in one sided-limits;

Also in the case $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

$\boxed{\text{Ex 7:}}$

$$\lim_{x\to\infty} \frac{5x + e^{-x}}{7x} = \frac{\infty + \frac{1}{e^{\infty}}}{7(\infty)} = \frac{\infty}{\infty}$$

$$\overset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{5 - e^{-x}}{7} = \boxed{\frac{5}{7}}$$

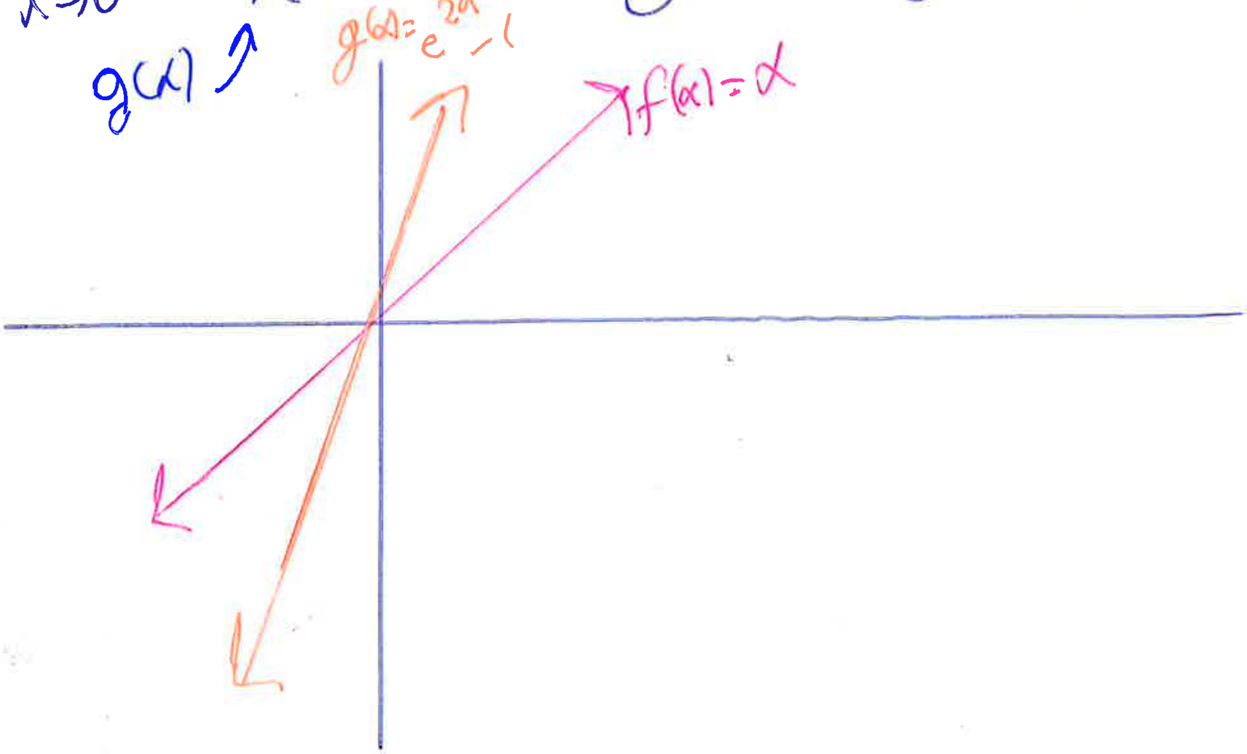
$\boxed{\text{Ex 3:}}$

$$\lim_{x\to 1} \frac{x^2-x}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$$

$$\lim_{x\to 1} \frac{x^2-x}{x-1} = \lim_{x\to 1} \frac{x\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x = 1$$

$\boxed{\text{Ex 4}}$

$$\lim_{x\to 0} \frac{\overbrace{e^{2x}-1}^{f(x)}}{\underbrace{x}_{g(x)}} = \frac{e^{2(0)}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$$



If we have two functions f and g are positive as $x \to \infty$, then we say g dominates over f

if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

$\lim_{x \to \infty} \frac{x}{e^x}$; first look at $x = 10{,}000$ $\frac{10{,}000}{e^{10{,}000}} \approx 0$

$\lim_{x \to \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$; use L'Hopital's

$\lim_{x \to \infty} \frac{1}{e^x} = 0$.