Monday, Oct. 31 - Fall '22 ①
Lecture #26

Announcements / Reminders
* Wiley Plus #9 due Wed (3.2, 3.3, some 3.4)
* Quiz 8 Thursday (same ↗)

Section 3.4 - The Chain Rule

How do we take the derivative of a nested function?

e^{x^2+x} $\quad (x^2+1)^{100}$ $\quad \sqrt{e^{(-x^3+1)}}$

These are compositions like $f(g(x))$
(Section 1.6)

How does $f(g(x))$ change when x changes a little bit?

The Chain Rule: $\frac{d}{dx}\Big(f(g(x))\Big) = \underbrace{f'(g(x))} \cdot \underbrace{g'(x)}$

take the derivative of the outside function (f), plug into it the inside function (g), not the deriv.

then multiply by the derivative of the inside function

* requires lots of practice *

Ex: $(x^2+1)^{100}$

$f(x)$ = outside function $= x^{100}$
$g(x)$ = inside function $= x^2+1$

check: $f(g(x))$
$= f(x^2+1)$
$= (x^2+1)^{100}$ ✓

$f'(x) = 100x^{99}$
$g'(x) = 2x$

$$f'(g(x)) \cdot g'(x) = f'(x^2+1) \cdot 2x$$
$$= 100(x^2+1)^{99} \cdot 2x$$
$$= 200x(x^2+1)^{99}$$

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x)$$

One way to think about this is:

- pretend the inside function is just x and take the derivative
- then multiply by the derivative of the inside.

Ex: $(x^2+1)^{100}$

$$(\quad)^{100}$$

$$\rightarrow 100 \cdot (\quad)^{99} \cdot \frac{d}{dx}(\quad)$$

$$\rightarrow 100 \cdot (x^2+1)^{99} \cdot (2x)$$

$$= 200x\,(x^2+1)^{99}$$

Ex: $e^{(x^2)}$

outside $= f(x) = e^x$

inside $= g(x) = x^2$

check $f(g(x)) = f(x^2) = e^{(x^2)}$

If we had it backward $f(x)=x^2$

$g(x)=e^x$

$$f(g(x)) = f(e^x) = (e^x)^2 \neq e^{(x^2)}$$

$$(e^x)^2 = e^{2x}$$

$f(x)=e^x \qquad g(x)=x^2$

$f'(x)=e^x \qquad g'(x)=2x$

$$f'(x^2)\cdot(2x) = e^{(x^2)}\cdot 2x$$

Other way: e^{x^2}

$$e^{\bigcirc}\cdot\frac{d}{dx}\left(\bigcirc\right)$$

$$e^{(x^2)}\cdot\frac{d}{dx}(x^2) = e^{(x^2)}\cdot 2x \quad \checkmark$$

Sometimes you need to apply the chain rule multiple times.

Ex: $\left(e^{-x/7}+5\right)^{1/2}$

$f(x)=x^{1/2}$
$g(x)=e^{-x/7}+5$

$f'(x)=\frac{1}{2}x^{-1/2}$

$g'(x)=$ we don't know yet because we need the chain rule to find the deriv. of $e^{-x/7}$.

$-\frac{1}{7}\cdot x$

$\frac{d}{dx}\left(e^{-x/7}\right)$

inside $= m(x) = -x/7$
outside $= n(x) = e^{x}$

$n(m(x)) = e^{-x/7}$ ✓

$n'(x) = e^{x}$
$m'(x) = -1/7$

By the chain rule:

$n'(m(x))\cdot m'(x) = n'\left(-\frac{x}{7}\right)\cdot\left(-\frac{1}{7}\right)$

$e^{-x/7}\cdot\left(-\frac{1}{7}\right)$

$g'(x) = -\frac{1}{7}e^{-x/7}+0$

$f(x)=x^{1/2}$
$g(x)=e^{-x/7}+5$

$f'(x)=\frac{1}{2}x^{-1/2}$

$$\frac{d}{dx}\left(e^{-x/7}+5\right)^{1/2}$$

$$= f'(g(x))\cdot g'(x)$$

$$= f'\left(e^{-x/7}+5\right)\cdot\left(-\frac{1}{7}e^{-x/7}\right)$$

$$= \underbrace{\frac{1}{2}\left(e^{-x/7}+5\right)^{-1/2}}_{f'(g(x))}\cdot\underbrace{\left(-\frac{1}{7}e^{-x/7}\right)}_{g'(x)}$$

$$= -\frac{e^{-x/7}}{14\sqrt{e^{-x/7}+5}}$$

Cloud Method:

$$\frac{d}{dx}\left(\left(e^{-x/7}+5\right)^{1/2}\right)$$

$$= \frac{1}{2}\bigcirc^{-1/2}\cdot\frac{d}{dx}\bigcirc$$

$$= \frac{1}{2}\left(e^{-x/7}+5\right)^{-1/2}\cdot\left(\frac{d}{dx}\left(e^{-x/7}\right)+\cancelto{0}{\frac{d}{dx}(5)}\right)$$

we need another cloud

$$= \frac{1}{2}\left(e^{-x/7}+5\right)^{-1/2}\cdot e^{\bigcirc}\cdot\frac{d}{dx}\left(\bigcirc\right)$$

$$= \frac{1}{2}\left(e^{-x/7}+5\right)^{-1/2}\cdot e^{-x/7}\cdot\frac{d}{dx}\left(-\frac{x}{7}\right)$$

$$= \frac{1}{2}\left(e^{-x/7}+5\right)^{-1/2}\cdot e^{-x/7}\cdot\left(-\frac{1}{7}\right)$$

Triple Chain Rule:

$$\frac{d}{dx}\left(\sqrt{e^{\sqrt{3+x^2}}}\right)$$

outermost $= \sqrt{x}$

next $= e^{x}$

next $= \sqrt{3+x^2}$

$$\frac{1}{2}\, e^{\sqrt{3+x^2}\ \,-1/2} \cdot \frac{d}{dx}\, e^{\sqrt{3+x^2}}$$

$$= \frac{1}{2}\left(e^{\sqrt{3+x^2}}\right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{d}{dx}\sqrt{3+x^2}$$

$$= \frac{1}{2}\left(e^{\sqrt{3+x^2}}\right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{2}\left(3+x^2\right)^{-1/2} \cdot \frac{d}{dx}\left(3+x^2\right)$$

$$= \frac{1}{2}\left(e^{\sqrt{3+x^2}}\right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{2}\left(3+x^2\right)^{-1/2} \left(2x\right)$$