Friday, Oct. 28 - Fall '22
Lecture #25

Announcements / Reminders

* Wiley Plus #9 due next Wed (3.2, 3.3, some 3.4)
* Quiz 8 next Thursday (same)

Section 3.3 - The Product and Quotient Rules

In 3.1: $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

WARNING: $\frac{d}{dx}(f(x)\cdot g(x)) \neq f'(x)\cdot g'(x)$

Formula #1: The product rule

$$\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)g(x)+f(x)g'(x)$$

Sometimes written as:

$$\frac{d}{dx}(\underbrace{uv}_{\text{implicity functions of } x}) = \frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}$$

In English: ②

"(derivative of the first) times the second + the first times (derivative of the second)"

first $= f(x)$

second $= g(x)$

$$\frac{d}{dx}(a^x) = \ln(a)\cdot a^x$$

Ex: Let $h(x) = \underbrace{x^2}_{f(x)} \cdot \underbrace{e^x}_{g(x)}$. Find $h'(x)$.

$f'(x) = 2x$

$g'(x) = e^x$

$$h'(x) \neq f'(x)g'(x)$$

$$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$= (2x)\cdot e^x + (x^2)\cdot e^x$$

$$= (2x+x^2)e^x = x(2+x)e^x$$

Ex: Find $h''(x)$.

We need to find the derivative of

$$\underbrace{(2x+x^2)}_{f(x)}\underbrace{e^x}_{g(x)}.$$

$f'(x) = 2+2x$

$g'(x) = e^x$

$h'' = f' \cdot g + f \cdot g'$ $\quad f \cdot g' + f' \cdot g$

$f \cdot g' + g \cdot f'$

$= (2+2x)e^x + (2x+x^2)e^x$

$= (2+4x+x^2)e^x$ ✓

Ex: $(x^2+1) \cdot 2^x \cdot \sqrt{x}$ Two ways

Combine $\sqrt{x} \cdot (x^2+1) = x^{1/2} \cdot (x^2+1)$

$= x^{1/2} \cdot x^2 + x^{1/2} \cdot 1$

$= x^{5/2} + x^{1/2}$

So, $(x^2+1) \cdot 2^x \cdot \sqrt{x} = \overbrace{(x^{5/2}+x^{1/2})}^{f(x)} \cdot \overbrace{2^x}^{g(x)}$

Use the regular product rule.

$f'(x) = \frac{5}{2}x^{3/2} + \frac{1}{2}x^{-1/2}$

$g'(x) = \ln(2) \cdot 2^x$

$f' \cdot g + f \cdot g' =$

$\left(\frac{5}{2}x^{3/2} + \frac{1}{2}x^{-1/2}\right) \cdot 2^x + \left(x^{5/2}+x^{1/2}\right) \cdot \ln(2) \cdot 2^x$

$$\frac{d}{dx}(u\cdot v\cdot w) = \frac{d}{dx}\big(\overbrace{u}^{f} \cdot \overbrace{(v\cdot w)}^{g}\big)$$ (4)

$$= u'\cdot(v\cdot w) + u\cdot(v\cdot w)'$$

$$= u'\cdot(v\cdot w) + u\,\underbrace{(v'w + v\cdot w')}$$

Ex: $\frac{e^x}{x^2}$ make this look like a product

$$= \boxed{e^x}\cdot\boxed{x^{-2}} \qquad \frac{1}{x^2} = x^{-2}$$

$f(x)$ $g(x)$

$f'(x) = e^x$ $g'(x) = (-2)x^{-3}$ $e^x(x^{-2} - 2x^{-3})$

Derivative $= e^x x^{-2} + e^x(-2x^{-3})$

$$\frac{e^x}{x^2}\left(1 - \frac{2}{x}\right) \qquad = e^x\left(\frac{1}{x^2} - \frac{2}{x^3}\right)$$

$$= \frac{e^x}{x^3}(x - 2)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - u\cdot\frac{dv}{dx}}{v^2}$$

$f(x)$ = "high" $\quad$ $g(x)$ = "low"

"low d-high minus high d-low over low-low"

$g(x)\ f'(x)\ -\ f(x)\cdot g'(x)$

$(g(x))^2$

Ex: $\frac{5x^2}{x^3+1}$ $\quad \frac{f(x)}{g(x)}$

$5x^2(x^3+1)^{-1}$ product rule doesn't work because we don't know how to take the deriv. of "$(x^3+1)^{-1}$" (yet)

$f'(x) = 10x$

$g'(x) = 3x^2$

$$\frac{(10x)(x^3+1) - (5x^2)(3x^2)}{(x^3+1)^2}$$

$$= \frac{(10x^4 + 10x) - (15x^4)}{(x^3+1)^2}$$

$$= \frac{-5x^4 + 10x}{(x^3+1)^2} = \frac{5x(-x^3+2)}{(x^3+1)^2}$$

Ex: $\frac{1}{1+e^x}$ $f(x) = 1$ $f'(x) = 0$

$g(x) = 1 + e^x$ $g'(x) = e^x$

$$\frac{0 \cdot \cancel{(1+e^x)} - 1 \cdot e^x}{(1+e^x)^2}$$

$$-\frac{e^x}{(1+e^x)^2}$$

Group Work:

#1) $\underbrace{x^5}_{f(x)} \cdot \underbrace{e^x}_{g(x)}$

$(5x^4)e^x + (x^5)e^x$

$= x^4 e^x (5 + x)$

#2) $\frac{t^2+1}{t+1}$) $f(t)$) $g(t)$

$$\frac{(t+1)(2t) - (t^2+1)(1)}{(t+1)^2}$$

$$= \frac{t^2 + 2t - 1}{(t+1)^2}$$