Homework 9+10

Math 2100/2105/2350
Fall 2018 $\boxed{1}$

HW9

1) Prove that there exists a positive integer n such that

$$\frac{1}{n\ln(n)} < 0.0001.$$

Proof by contradiction:

ATAC that $\frac{1}{n\ln(n)} \geq 0.0001$ for all $n \in \mathbb{N}^{+}$. Note that $0.0001 = \frac{1}{10000}$.

If $\frac{1}{n\ln(n)} \geq \frac{1}{10000}$, then by cross-multiplying,

$$10000 \geq n\ln(n).$$

But $\ln(n) > 1$ for all $n \geq 3$, so when $n = 10000$

$$10000 \cdot \ln(10000) \geq 10000,$$

but our assumption says

$$10000 \cdot \ln(10000) \leq 10000.$$

This is a contradiction. So, our assumption was wrong, and there does exist a positive integer n such that

$$\frac{1}{n\ln(n)} < 0.0001$$

2) [graded question]

3) Prove that if $a+b+c+d \geq 26$, then either $a \geq 3$, $b \geq 7$, $c \geq 7$, or $d \geq 9$.

<u>Proof by contradiction:</u>

Suppose by way of contradiction that $a+b+c+d \geq 26$ but not $\underbrace{(a \geq 3, b \geq 7, c \geq 7, \text{ or } d \geq 9)}$.

The negation of this is
$a < 3$, $b < 7$, $c < 7$, <u>and</u> $d < 9$.

This would imply $a+b+c+d < 26$, a contradiction. ∎

4) Use the PHP to prove that given any five integers, there will be two that have a sum or difference divisible by 7.

<u>Proof:</u> Define four buckets depending on the remainder of the # when divided by 7.

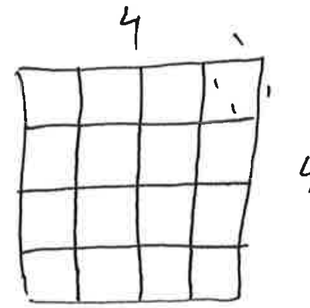
By the PHP, given any five integers, at least two of them, say k and l, are in the same bucket. k and l either have the same remainder mod 7, or different. If they have the same, then $k = 7M+c$ and $l = 7N+c$ for some $M, N \in \mathbb{Z}$. Then $k-l = 7(M-N)$, so their difference is div. by 7. If they have different remainders, then by the construction of the buckets, the remainders add up to 7. So, k has the form $k = 7M+c$, and $l = 7N+d$. Thus $k+l = 7(M+N) + (c+d) = 7(M+N)+7 = 7(M+N+1)$, and so their sum is div. by 7. ∎

5) [graded question]

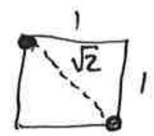
6) Show that if you pick 17 points from a square with side length 4, then there must be 2 of those points that are within $\sqrt{2}$ of each other.

<ins>Proof:</ins>

Split the square into 16 parts.



By the PHP, one of these squares will have at least two points.



The furthest distance between two points in this square is $\sqrt{2}$.

HW 10

1) [graded question]

2) Prove or disprove: For any two sets A and B, $A \setminus B = A \cap \overline{B}$.

<ins>Proof:</ins>

<ins>Part 1:</ins> $A \setminus B \subseteq A \cap \overline{B}$.

Let $x \in A \setminus B$. This implies that $x \in A$ and $x \notin B$.

Since $x \notin B$, we know $x \in \overline{B}$. Thus $x \in A \cap \overline{B}$.

Hence $A \setminus B \subseteq A \cap \overline{B}$.

<ins>Part 2:</ins> $A \cap \overline{B} \subseteq A \setminus B$

Let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$. Since $x \in \overline{B}$, we have $x \notin B$. Thus $x \in A \setminus B$.

3) [graded question]

4) Prove that

$$\left(\{6k+1 : k \in \mathbb{Z}\} \cup \{6m-1 : m \in \mathbb{Z}\} \right) \subseteq \{2n+1 : n \in \mathbb{Z}\}.$$

Let $\ell \in$ LHS of $\nearrow$. Then either $\ell \in \{6k+1 : k \in \mathbb{Z}\}$ or $\ell \in \{6m-1 : m \in \mathbb{Z}\}$ (or both).

Case 1: $\ell \in \{6k+1 : k \in \mathbb{Z}\}$

Then, $\ell = 6K+1$ for some $K \in \mathbb{Z}$.

So, $\ell = 2(3K)+1$. Since $3K \in \mathbb{Z}$, $\ell \in \{2n+1 : n \in \mathbb{Z}\}$.

Case 2: $\ell \in \{6m-1 : m \in \mathbb{Z}\}$

Then, $\ell = 6M-1$ for some $M \in \mathbb{Z}$.

So, $\ell = 2(3M-1)+1$.

Since $3M-1 \in \mathbb{Z}$, we have $\ell \in \{2n+1 : n \in \mathbb{Z}\}$.

5) Use induction to prove that $|P(A)| = 2^{|A|}$.

$P(n) =$ "If $|A| = n$, then $|P(A)| = 2^n$."

Base case, $n=0$. If $|A| = 0$, then $A = \emptyset$ and $P(A) = \{\emptyset\}$,

so $|P(A)| = 1$

$2^{|A|} = 2^0 = 1$ ✓

Induction Case:

Assume that $P(n-1) =$ "If $|A| = n-1$ then $|P(A)| = 2^{n-1}$" is true.

Let S be a set of size n. Pick one special element $x \in S$.

Let $T = S \setminus \{x\}$. Then $|T| = n-1$. By the induction hypothesis, There are 2^{n-1} subsets of T.

Every subset of S either $\underbrace{\text{has } x}$ or $\underbrace{\text{doesn't}}$.

The subsets of S that don't have x are the subsets of T.
The subsets of S that do have x are exactly the subsets of T plus x

So, the total # of subsets is $2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$. ∎

subsets of $\{1,2,3\}$

don't have 3	have 3
$\{\}$	$\{3\}$
$\{1\}$	$\{13\}$
$\{2\}$	$\{23\}$
$\{1,2\}$	$\{123\}$.