# Homework 8

Math 2100/2105/2350
Fall 2018

$\boxed{11}$

1) Prove that for all $n \in \mathbb{Z}$ with $n \geq 4$, $n! > 2^n$.

Proof by induction:

Define the predicate $P(n) =$ "$n! > 2^n$".

The base case is $P(4)$, which says $4! > 2^4$.

$4! = 24$ and $2^4 = 16$. Since $24 > 16$, the base case holds.

For the induction step, assume $k \in \mathbb{N}$, $k \geq 5$ is arbitrary and assume that

$$P(k-1) = \text{“}(k-1)! > 2^{k-1}\text{”}$$

is true. Our goal is to prove that

$$P(k) = \text{“}k! > 2^k\text{”}$$

is true. Observe that

$$k! = k\cdot(k-1)! > k\cdot 2^{k-1} > 2\cdot 2^{k-1} = 2^k$$

def. of "!"

induction hyp. and arithmetic: if $a > b$ and $c > 0$ then $a \cdot c > b \cdot c$

since $k > 2$ and

arithmetic.

This shows that $k! > 2^k$, and so the induction step holds. $\boxed{2}$

Thus, by induction, $P(n)$ is true for all $n \geq 4$.

2) Prove that for all $n \in \mathbb{Z}^+$, $5^{2n+1} + 2^{2n+1}$ is div. by 7.

<u>Proof by induction:</u>

Define the predicate

$$P(n) = \text{"}5^{2n+1} + 2^{2n+1} \text{ is divisible by } 7\text{"}$$

The base case is $P(1) = $ "$5^3 + 2^3$ is div. by 7."

$$5^3 + 2^3 = 125 + 8 = 133 = 7 \cdot 19,$$

and so the base case holds.

Let $k \in \mathbb{N}$, $k \geq 2$ be arbitrary. Assume that $P(k-1) = $ "$5^{2k-1} + 2^{2k-1}$ is div. by 7" is true. We want to show that $P(k) = $ "$5^{2k+1} + 2^{2k+1}$ is div. by 7."

Arithmetic shows that

$$\begin{aligned}
5^{2k+1} + 2^{2k+1} &= 5^2 \cdot 5^{2k-1} + 2^2 \cdot 2^{2k-1} \\
&= 25 \cdot 5^{2k-1} + 4 \cdot 2^{2k-1} \\
&= (21+4) \cdot 5^{2k-1} + 4 \cdot 2^{2k-1} \\
&= \underbrace{21 \cdot 5^{2k-1}}_{A} + \underbrace{4 \cdot (5^{2k-1} + 2^{2k-1})}_{B}
\end{aligned}$$

The A term is divisible by 7 because 21 is divisible by 7, which implies $A = 7 \cdot \underbrace{(3 \cdot 5^{2k-1})}_{\substack{\text{integer because} \\ k \geq 2}}$.

The B term is divisible by 7 because the induction hypothesis tells us that $5^{2k-1} + 2^{2k-1}$ is divisible by 7.

By a theorem from ~~Homework 7~~ Quiz 6, if 7 divides A and 7 divides B, then 7 divides A+B, which is what we wanted to show.

Thus, by induction, P(n) holds for all $n \geq 1$. ∎

3) [graded question]

4) Prove that for all $n \in \mathbb{Z}^+$,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

You may use the fact that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof by induction:

Define $P(n) =$ "$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$".

The base case is $P(1) =$ "$1^3 = (1)^2$" which is clearly true.

For the induction step, suppose $k \in \mathbb{Z}^+$, $k \geq 2$ is arbitrary.

We can assume that

$$P(k-1) = \text{"}1^3 + 2^3 + \dots + (k-1)^3 = (1+2+\dots+(k-1))^2\text{"}$$

is true, and we want to prove that

$$P(k) = \text{"}1^3 + 2^3 + \dots + k^3 = (1+2+\dots+k)^2\text{"}$$

is true.

By arithmetic:

$$\begin{aligned}
1^3+2^3+\dots+k^3 &= (1^3+2^3+\dots+(k-1)^3) + k^3 && \text{(grouping)} \\
&= (1+2+\dots+(k-1))^2 + k^3 && \text{(by induction hypothesis)} \\
&= \left(\frac{(k-1)(k)}{2}\right)^2 + k^3 && \text{(by the fact we can use)} \\
&= \frac{(k^2-k)^2}{4} + k^3 \\
&= \frac{k^4-2k^3+k^2}{4} + k^3 \\
&= \frac{k^4-2k^3+k^2+4k^3}{4} \\
&= \frac{k^4+2k^3+k^2}{4} \\
&= \frac{k^2(k^2+2k+1)}{4} \\
&= \frac{k^2(k+1)^2}{4} = \left(\frac{k(k+1)}{2}\right)^2 = (1+2+\dots+k)^2.
\end{aligned}$$

by arithmetic

by the fact we can assume.

So, by induction, $P(n)$ holds for all $n \geq 1$. $\boxed{\Xi}$