Homework 7

Math 2100/2105/2350
Fall 2018

1) Prove that for any integer n, if n^2 is odd, then n is odd.

Proof by ~~contradiction~~ contrapositive:

The contrapositive is:

"If n is even, then n^2 is even."

Suppose n is an arbitrary even integer.
By the definition of "even", $n = 2k$ for some $k \in \mathbb{Z}$.
Thus, $n^2 = (2k)^2 = 4k^2 = 2\underbrace{(2k^2)}$.
Since $k \in \mathbb{Z}$, we have $2k^2 \in \mathbb{Z}$ as well.
Thus n^2 is 2 times an integer, which implies that n^2 is even. ∎

2) [graded question]

3) Is it true that for all $m \in \mathbb{Z}$, 3 divides $m^3 - m$?
Yes.

Proof: (by cases)

Let $m \in \mathbb{Z}$ be arbitrary.
Observe that $m^3 - m = m(m^2 - 1) = (m-1) \cdot m \cdot (m+1)$.

Case 1: m is divisible by 3.

If m is divisible by 3, then by Exercise 2,

since 3 divides m and m divides $(m-1)m(m+1)$,

we conclude 3 divides $(m-1)m(m+1)$.

Case 2: m is one more than a multiple of 3.

If m is one more than a multiple of 3, then $m-1$ is a multiple of 3.

By the same logic Case 1, this implies 3 divides $(m-1)m(m+1)$.

Case 3: m is two more than a multiple of 3.

If m is two more than a multiple of 3, then $m+1$ is a multiple of 3.

By the same logic as Case 1, this implies 3 divides $(m-1)m(m+1)$.

By the Division Theorem, these are the only three cases that need to be checked.

4) Prove that all integers are rational numbers.

Proof:

Let n be an arbitrary integer.

Note that $n = \frac{n}{1}$.

The def. of the rationals is

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Since $n \in \mathbb{Z}$, $1 \in \mathbb{Z}$, and $1 \neq 0$, we conclude that $\frac{n}{1} \in \mathbb{Q}$. $\blacksquare$

5) [graded question]

6) True or False: If n is a positive even integer, $n \geq 4$ then $2^n - 1$ is not prime.

True!

Proof: Let n be an arbitrary positive even integer

Then, $n = 2k$ for some $k \in \mathbb{Z}$.

Thus, $2^n - 1 = 2^{2k} - 1 = (2^k)^2 - (1)^2 = (2^k + 1)(2^k - 1)$

(difference of squares).

Since $n \geq 4$, we know $k \geq 2$, and thus $2^k - 1 \geq 3$.

This proves that $2^n - 1$ is not prime because it factors into two integers, each > 1. $\blacksquare$