MATH 2100 / 2105 / 2350 – EXAM 2

November 15, 2018

Name: Key

Instructions: Please write your work neatly and clearly. **You must explain all reasoning. It is not sufficient to just write the correct answer.** You have 75 minutes to complete this exam. You may not use calculators, notes, or any other external resources.

Scores

1	
2	
3	
4	
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The Marquette University honor code obliges students:

- To fully observe the rules governing exams and assignments regarding resource material, electronic aids, copying, collaborating with others, or engaging in any other behavior that subverts the purpose of the exam or assignment and the directions of the instructor.
- To turn in work done specifically for the paper or assignment, and not to borrow work either from other students, or from assignments for other courses.
- To complete individual assignments individually, and neither to accept nor give unauthorized help.
- To report any observed breaches of this honor code and academic honesty.

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1. Let A and B be sets. Define the following things. You must give both their names and their definitions.

   (a) $A \cap B$

   "A intersect B" is the set of elements in both A and B

   (b) $A \times B$

   "A cross B" or "A cartesian product B" is the set of ordered pairs (a,b) such that $a \in A$ and $b \in B$.

   (c) $\mathcal{P}(A)$

   The "powerset of A" is the set of all subsets of A.

2. Prove that for any positive integer n, if n^2 is odd then n is odd.

Proof by Contrapositive:

Suppose n is an even positive integer.
Then, $n = 2k$ for some $k \in \mathbb{Z}$.
Thus $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Since $2k^2 \in \mathbb{Z}$ (because $k \in \mathbb{Z}$), we have that n^2 is even. ∎

3. Define the predicate $Z(a,b,c) =$ "$a^2 + b^2 = c^2$". Determine whether each of the following are true, and justify your reasoning.

   (a) $\exists a \in \mathbb{N}, \exists b \in \mathbb{N}, \exists c \in \mathbb{N}, Z(a,b,c)$

True: $Z(0,0,0) =$ "$0^2+0^2=0^2$", which is true.

   (b) $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \exists c \in \mathbb{R}, Z(a,b,c)$

True: Given a fixed $a, b \in \mathbb{R}$, define $Q = \sqrt{a^2+b^2}$ (exists because $a^2+b^2 \geq 0$).

Then $Z(a,b,Q) =$ "$a^2+b^2 = (\sqrt{a^2+b^2})^2$", which is true.

   (c) $\forall a \in \mathbb{Q}, Z(a, 2a, 3a)$

False. Consider $a=1$.

Then $Z(1,2,3) =$ "$1^2+2^2=3^2$" which is false.

4. Prove by contradiction that there do not exist integers a and b such that $18a + 6b = 1$.

Proof:

ATAC that $\exists a, b \in \mathbb{Z}$ such that $18a + 6b = 1$.

Then, dividing both sides by 6, we see

$$3a + b = \frac{1}{6}.$$

But, the LHS is an integer and the RHS is not, which is a contradiction. $\blacksquare$

5. Prove or disprove that for any two sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof:

Let $X \in P(A) \cup P(B)$ be arbitrary.

Then either $X \in P(A)$ or $X \in P(B)$.

By the definition of power set,

$X \subseteq A$ or $X \subseteq B$.

Therefore $X \subseteq A \cup B$.

By the definition of power set again,

$X \in P(A \cup B)$.

We've proved that if $X \in P(A) \cup P(B)$ then $X \in P(A \cup B)$.

Thus $P(A) \cup P(B) \subseteq P(A \cup B)$. ∎

6. Prove that for all $n \geq 0$:

$$\sum_{i=0}^{n} i2^i = 2 + (n-1)2^{n+1}$$

Proof by induction:

Define $P(n) =$ "$\sum_{i=0}^{n} i2^i = 2 + (n-1)2^{n+1}$".

The base case is

$$P(0) = \text{"}\sum_{i=0}^{0} i \cdot 2^i = 2 + (-1)2\text{"}$$

$$= \text{"}0 = 0\text{"}, \text{ which is true.}$$

For the induction step, assume $P(k-1) =$ "$\sum_{i=0}^{k-1} i2^i = 2 + (k-2)2^k$" is true. Then

$$\sum_{i=0}^{k} i2^i = \left[\sum_{i=0}^{k-1} i2^i\right] + k2^k$$

by induction hypothesis

$$= \left(2 + (k-2)2^k\right) + k2^k$$

$$= 2 + k2^k - 2 \cdot 2^k + k \cdot 2^k$$

$$= 2 + 2^k(2k-2)$$

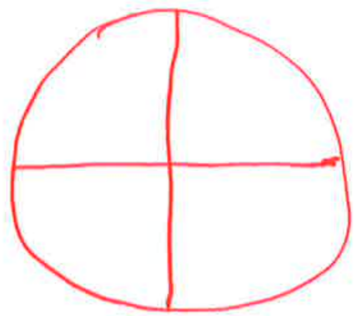
$$= 2 + 2^{k+1}(k-1).$$

This proves that $P(k)$ is true, completing the proof. ∎

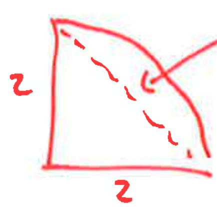
7. Prove that if any 5 points are chosen from a circle of radius 2, then there exist a pair of points within distance $\sqrt{8}$ of each other.

Proof:

Split the circle into 4 parts as follows:



By the PHP, given any 5 points, there must be at least 2 in the same sector. The furthest two points can be in a sector is



$\sqrt{2^2+2^2} = \sqrt{8}$.

So, the distance between those two points is at most $\sqrt{8}$. ∎

8. Let $A_1, A_2, A_3, \ldots$ be an infinite collection of sets with the property that $|A_1| = 1$, $|A_2| = 2$, $|A_3| = 3$, etc. In general, $|A_i| = i$. Prove (probably by induction) that for any $n \geq 1$:

$$|A_1 \times A_2 \times \cdots \times A_n| = n!.$$

You may use the fact that $|A \times B| = |A| \cdot |B|$.

Proof by induction:

Define $P(n) =$ " $|A_1 \times A_2 \times \ldots \times A_n| = n!$ "

The base case is $P(1) =$ " $|A_1| = 1!$ " which is true by assumption.

Now assume $P(k-1) =$ " $|A_1 \times \ldots \times A_{k-1}| = (k-1)!$ " is true.

So, $|A_1 \times \ldots \times A_k| = |(A_1 \times \ldots \times A_{k-1}) \times A_k|$

$= |A_1 \times \ldots \times A_{k-1}| \cdot |A_k|$ (by the fact)

by induction hypothesis $= (k-1)! \cdot k$

$= (k-1)!$.

So, $P(k)$ is true, completing the induction. ∎

9. (a) Prove that if 3 divides $4^{n-1} - 1$ then 3 divides $4^n - 1$.

Proof:

Suppose that 3 divides $4^{n-1} - 1$. Then, $4^{n-1} - 1 = 3k$ for some $k \in \mathbb{Z}$.

Now, observe that $4^{n-1} = 3k + 1$.

So, $4^n - 1 = 4 \cdot 4^{n-1} - 1 = 4(3k+1) - 1$

$= 12k + 4 - 1$

$= 12k + 3$

$= 3(4k+1)$.

Because $4^n - 1 = 3 \cdot (4k+1)$ and $4k+1 \in \mathbb{Z}$, we know that $4^n - 1$ is divisible by 3. $\blacksquare$

(b) What you proved in part (a) did not quite use induction, but it was close. Explain how your answer to part (a) would fit inside an induction proof of a more general fact.

This would be the induction step of the theorem: "For all $n \geq 1$, $4^n - 1$ is divisible by 3."

10. (a) Prove that the implication $p \to q$ is logically equivalent to $\neg p \vee q$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

These two columns are the same, so,

$p \to q \equiv \neg p \vee q$.

(b) The statement "If $\underbrace{n \text{ ends in } 5}_{p}$, then $\underbrace{n \text{ is a multiple of } 5}_{q}$" is written in the first form. Translate it to the second form.

$\neg p \vee q$ is "Either n does not end in 5 or n is a multiple of 5."