MATH 13, FALL '16

HOMEWORK 9

Will not be graded, just for practice.

Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. Consider the cube with opposite corner points $(0, 0, 0)$ and $(2, 2, 2)$. Let $\boldsymbol{F} = \langle -y, 2yz, 1 - z^2 \rangle$, and note that $\boldsymbol{F} = \text{curl}(\boldsymbol{A})$ for $\boldsymbol{A} = \langle yz^2, (yz + x), z^3 \rangle$.

   a) Let S be the surface of the cube (all six sides), with outward facing normal vectors. Calculate $\iint_S \boldsymbol{F}\, dS$.

   Solution: We can use Stokes' Theorem since $\boldsymbol{F}$ has a vector potential $\boldsymbol{A}$ (i.e., since $\boldsymbol{F}$ is the curl of something). [Note that without knowing that $\boldsymbol{A}$ exists, we can't apply Stokes' Theorem!]

   Stokes' Theorem says that $\iint_S \boldsymbol{F}\, dS$ is equal to the line integral of $\boldsymbol{A}$ around the boundary of S. Since the surface is closed, it has no boundary! Hence, we can immediately conclude that
   $$\iint \boldsymbol{F}\, dS = 0.$$

   b) Let S be the surface consisting of all sides except the top, with outward facing normal vectors. Calculate $\iint_S \boldsymbol{F}\, dS$.

   Solution: Like the last problem, we use Stokes' Theorem, but this time there is a non-empty boundary. The boundary is the boundary of the square with corners $(0, 0, 2)$, $(0, 2, 2)$, $(2, 0, 2)$, and $(2, 2, 2)$. We *could* use Stokes' Theorem just this once, but then we have to integrate along the boundary, which is four separate straight lines.

   Instead, let's try to apply Stokes' Theorem again (using the idea of *surface independence*) to change the integral over this boundary to the integral of some other surface with this boundary — hopefully, one that's easier than the 5-sided cube that we started with!

   The easiest such surface is P: the filled in square with corner points $(0, 0, 2)$, $(0, 2, 2)$, $(2, 0, 2)$, and $(2, 2, 2)$. But remember, all of our surfaces must be oriented. So, in addition to defining the new surface to be the square, we also need to know its orientation. We don't get to just pick, we have to make sure it's compatible with the normal vectors from the original surface. (Both surfaces must have the same *oriented boundary*.)

   Since the original cube has outward facing normal vectors, the boundary must be traversed in a clockwise direction when viewed from above. In order for the new surface (the flat square) to have the same boundary orientation, its normal vectors must be downward pointing.

At this point we have declared that the integral we want to calculate is equal to the integral over P (with downward pointing normal vectors) of $\boldsymbol{F}$. So, we proceed as in Section 16.5 to calculate this.

A good parametrization of P is

$$G(u,v) = \langle u, v, 2 \rangle$$

for $0 \le u \le 2$ and $0 \le v \le 2$. Then,

$$\begin{aligned} \boldsymbol{T}_u &= \langle 1, 0, 0 \rangle \\ \boldsymbol{T}_v &= \langle 0, 1, 0 \rangle \\ \boldsymbol{N} &= \langle 0, 0, 1 \rangle. \end{aligned}$$

Since we said we want downward pointing normal vectors, we use

$$\boldsymbol{N} = \langle 0, 0, -1 \rangle.$$

Now,

$$\begin{aligned} \iint_P \boldsymbol{F}\, dS &= \int_0^2 \int_0^2 \boldsymbol{F}(G(u,v)) \cdot \boldsymbol{N}\, du dv \\ &= \int_0^2 \int_0^2 \langle -v, 4v, -3 \rangle \cdot \langle 0, 0, -1 \rangle\, du dv \\ &= \int_0^2 \int_0^2 3\, du dv \\ &= 12. \end{aligned}$$

2. Let S be the surface consisting of the outside of the cone $z = \sqrt{x^2 + y^2}$ bounded by the planes $z = 1$ and $z = 4$, with inward facing normal vectors. Note that S *does not* include the top nor bottom surfaces, just the outside of the cone. Let $\boldsymbol{F} = \left\langle x + y, z - x, \cos(e^{x^2 - yz}) \right\rangle$. Calculate $\iint_S \operatorname{curl}(\boldsymbol{F})\, dS$.

   Solution: The messiness of Tuesday suggests that we should use Stokes' Theorem. This will help because the boundary of S consists of two circles: the circle parallel with the xy-plane with radius 1, centered around the z-axis, lying in $z = 1$ and the circle parallel with the xy-plane with radius 4, centered around the z-axis, lying in $z = 4$. We can see ahead that each of the parameterizations for these curves will lead to an $\boldsymbol{r}'(t)$ with a 0 in the z-component. When we take the dot product, this will get rid of the messy term.

   We have to check the orientation of the boundary. Since the cone has inward facing normal vectors, the circle in $z = 1$ should be traversed clockwise when viewed from above and the circle in $z = 4$ should be traversed counterclockwise when viewed from above.

   So, the boundary of the surface is two circles, parametrized by

$$\begin{aligned} \boldsymbol{r}_1(t) &= \langle \cos(t), -\sin(t), 1 \rangle \\ \boldsymbol{r}_2(t) &= \langle 4\cos(t), 4\sin(t), 4 \rangle \end{aligned}$$

for $0 \leq t \leq 2\pi$.

Now we integrate, $\boldsymbol{r}_1(t)$ first:

$$\begin{aligned}
\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}_1(t)) \cdot \boldsymbol{r}_1'(t)\,dt &= \int_0^{2\pi} \langle \cos(t) - \sin(t), 1 - \cos(t), \cos(e^{\cos^2(t)+\sin(t)}) \rangle \cdot \langle -\sin(t), -\cos(t), 0 \rangle \\
&= \int_0^{2\pi} (\sin^2(t) - \sin(t)\cos(t) - \cos(t) + \cos^2(t))\,dt \\
&= \int_0^{2\pi} (1 - \cos(t) - \cos(t)\sin(t))\,dt \\
&= \left[t - \sin(t) - \frac{\sin^2(t)}{2} \right]_0^{2\pi} \\
&= 2\pi.
\end{aligned}$$

Now $\boldsymbol{r}_2(t)$:

$$\begin{aligned}
\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}_1(t)) \cdot \boldsymbol{r}_1'(t)\,dt &= \int_0^{2\pi} \langle 4\cos(t) + 4\sin(t), 4 - 4\cos(t), \cos([\text{stuff}]) \rangle \cdot \langle -4\sin(t), 4\cos(t), 0 \rangle \\
&= \int_0^{2\pi} (-16\sin(t)\cos(t) - 16\sin^2(t) + 16\cos(t) - 16\cos^2(t))dt \\
&= \int_0^{2\pi} (-16 + 16\cos(t) - 16\sin(t)\cos(t))dt \\
&= \left[-16t + 16\sin(t) - 8\sin^2(t) \right]_0^{2\pi} \\
&= -32\pi.
\end{aligned}$$

Therefore,

$$\iint_S \operatorname{curl}(\boldsymbol{F})\,dS = 2\pi - 32\pi = -30\pi.$$

3. Let S be the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 2$ (so, S involves the outside of the cylinder as well as the top and bottom circular caps), with inward-facing normal vectors. Let $\boldsymbol{F} = \langle x^3, ze^x, 3zy^2 \rangle$. Find $\iint_S \boldsymbol{F}\,dS$.

Solution: First, note that Stokes' Theorem doesn't apply here because we don't know that $\boldsymbol{F}$ has a vector potential. If it did, the integral would automatically be zero.

Clearly, we want to use the Divergence Theorem, which says that

$$\iint_S \boldsymbol{F}\,dS = \iiint_W \operatorname{div}(\boldsymbol{F}),$$

where W is the *three-dimensional* region consisting of the interior of the cylinder. *Except*, the application of this theorem requires outward-facing normal vectors, while ours are inward. Therefore, when we apply the Divergence Theorem we have to add a negative sign:

$$\iint_S \boldsymbol{F}\,dS = -\iiint_W \operatorname{div}(\boldsymbol{F}).$$

So,

$$\begin{aligned}
\iint_S \boldsymbol{F}\, dS &= -\iiint_W \operatorname{div}(\boldsymbol{F}) \\
&= -\iiint_W \left(3x^2 + 3y^2\right) dV \\
&= -\int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2(\theta) + 3r^2\sin^2(\theta))r\, dxdrd\theta \\
&= -\int_0^{2\pi}\int_0^1\int_{-1}^2 3r^3\, dxdrd\theta \\
&= -3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r^3\, dr\right)\left(\int_{-1}^2 dz\right) \\
&= -3(2\pi)\left[\frac{r^4}{4}\right]_0^1 (3) \\
&= -\frac{9\pi}{2}.
\end{aligned}$$

4. Let W be the part of the solid ball $x^2 + y^2 + z^2 \leq 4$ with $z \geq 0$. Let $S = \partial W$ be the boundary surface of W, oriented with outward pointing normal vectors. The surface S consists of two smooth pieces: $S = S_1 + S_2$, where S_1 is the disk $x^2 + y^2 \leq 4$, $z = 0$; and S_2 is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. Finally, we have a vector field

$$\mathbf{F} = \langle y^3 + z^3,\ x^5 + z^5,\ x^2 + y^2\rangle$$

a) What is the value of $\iint_S \mathbf{F}\, dS$?

Solution: First observe that

$$\operatorname{div}(\boldsymbol{F}) = 0.$$

So, by the Divergence Theorem,

$$\iint_S \boldsymbol{F} dS = \iiint_W \operatorname{div}(\boldsymbol{F}) dV = \iiint_W 0\, dV = 0.$$

b) What is the value of $\iint_{S_1} \mathbf{F}\, dS$?

Solution: We will compute this surface integral directly. We'll use the parametrization of the disk

$$G(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle.$$

From this we calculate

$$\begin{aligned}
\boldsymbol{T}_r &= \langle \cos(\theta), \sin(\theta), 0\rangle \\
\boldsymbol{T}_\theta &= \langle -r\sin(\theta), r\cos(\theta), 0\rangle \\
\boldsymbol{N} &= \langle 0, 0, r\rangle.
\end{aligned}$$

This is an inward-pointing normal vector, but we want an outward-pointing normal vector, so we must negate it:

$$\boldsymbol{N} = \langle 0, 0, -r\rangle.$$

Hence,

$$\begin{aligned}
\iint_{S_1} \boldsymbol{F}\, dS &= \int_0^{2\pi} \int_0^2 \boldsymbol{F}(G(r,\theta)) \cdot \boldsymbol{N}\, dr d\theta \\
&= \int_0^{2\pi} \int_0^2 \langle [\text{stuff}], [\text{stuff}], r^2 \rangle \cdot \langle 0, 0, -r \rangle\, dr d\theta \\
&= \int_0^{2\pi} \int_0^2 -r^3\, dr d\theta \\
&= -2\pi \left[\frac{r^4}{4} \right]_0^2 \\
&= -8\pi.
\end{aligned}$$

c) What is the value of $\iint_{S_2} \mathbf{F},\, dS$?

Solution: By the additivity of surfaces:

$$\iint_S \boldsymbol{F}\, dS = \iint_{S_1} \boldsymbol{F}\, dS + \iint_{S_2} \boldsymbol{F}\, dS.$$

Since $\iint_S \boldsymbol{F}\, dS = 0$ by part (a), it follows that

$$\iint_{S_2} \boldsymbol{F}\, dS = -\iint_{S_1} \boldsymbol{F}\, dS = -(-8\pi) = 8\pi.$$