MATH 13, FALL '16

HOMEWORK 8

Due Wednesday Nov 9

Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. Use surface integrals to find the integral of $f(x,y,z) = x + z^2$ of the cylinder given by $x^2 + y^2 = 4$ and $-2 \leq z \leq 2$, including the top and bottom caps of the cylinder. (You should need three surface integrals. Think very carefully about your answers to each part to check if they make sense. For example: "If I integrate this function on these points, should my answer be positive or negative?".)

   Solution: We need to parametrize three surfaces: the curved part of the cylinder and the top and bottom. The curved part is parametrized (as we saw in class) by

   $$G_1(u,v) = \langle 2\cos(v), 2\sin(v), u\rangle,$$

   for $-2 \leq u \leq 2$ and $0 \leq v \leq 2\pi$.

   The top and bottom are parametrized by

   $$\begin{aligned} G_2(u,v) &= \langle u\cos(v), u\sin(v), 2\rangle, \\ G_3(u,v) &= \langle u\cos(v), u\sin(v), -2\rangle, \end{aligned}$$

   for $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$.

   Now we need to find the normal vectors for each. The tangent vectors for G_1 are:

   $$\begin{aligned} \boldsymbol{T_{u,1}} &= \langle 0, 0, 1\rangle, \\ \boldsymbol{T_{v,1}} &= \langle -2\sin(v), 2\cos(v), 0\rangle, \end{aligned}$$

   so the normal vector for G_1 is

   $$\boldsymbol{N_1} = \boldsymbol{T_{u,1}} \times \boldsymbol{T_{v,1}} = \langle -2\cos(v), -2\sin(v), 0\rangle,$$

   and therefore

   $$\|\boldsymbol{N_1}\| = 2.$$

   The tangent vectors for G_2 are:

   $$\begin{aligned} \boldsymbol{T_{u,2}} &= \langle \cos(v), \sin(v), 0\rangle, \\ \boldsymbol{T_{v,2}} &= \langle -u\sin(v), u\cos(v), 0\rangle, \end{aligned}$$

so the normal vector for G_2 is

$$\boldsymbol{N_2} = \boldsymbol{T_{u,2}} \times \boldsymbol{T_{v,2}} = \langle 0, 0, u \rangle,$$

and therefore

$$\|\boldsymbol{N_2}\| = |u| = u.$$

Note that $|u| = u$ because for G_2 and G_3 the bounds are $0 \le u \le 2$.

The tangent vectors for G_3 are the same as for G_2! So $\|\boldsymbol{N_3}\| = u$.

Now we need to do three integrals.

$$\begin{aligned} I_1 &= \int_{-2}^{2} \int_0^{2\pi} ((2\cos(v) + u^2) \cdot 2)\, dv du \\ &= \dots \\ &= \frac{64\pi}{3}. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^2 \int_0^{2\pi} ((u\cos(v) + 4) \cdot u)\, dv du \\ &= \dots \\ &= 16\pi \end{aligned}$$

$$\begin{aligned} I_3 &= \int_0^2 \int_0^{2\pi} ((u\cos(v) + 4) \cdot u)\, dv du \\ &= \dots \\ &= 16\pi \end{aligned}$$

Therefore, $\iint_S (x + z^2)\, dS = \frac{64\pi}{3} + 16\pi + 16\pi = \frac{160\pi}{3}$.

2. Compute $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ for $\boldsymbol{F} = \langle 0, 0, z + y \rangle$ where S is the cone $2x^2 + 2y^2 = z^2$ between $z = 0$ and $z = 1$, with upward pointing normal vector.

   Solution: The trickiest part here is probably parametrizing the cone. In class we dealt with a similar cone $x^2 + y^2 = z^2$, and we found the parametrization

   $$H(u, v) = \langle u\cos(v), u\sin(v), u \rangle.$$

   Our cone is similar, but with steeper edges. Here's how we can find the correct parametrization. Note that if $2x^2 + 2y^2 = z^2$ then

   $$z = \sqrt{2x^2 + 2y^2}.$$

   If we're going to set $x = u\cos(v)$ and $y = y\sin(v)$ like the regular cone, then the z coordinate should be

   $$z = \sqrt{2u^2\cos^2(v) + 2u^2\sin^2(v)} = u\sqrt{2}.$$

So, we'll use the parametrization

$$G(u,v) = \langle u\cos(v), u\sin(v), u\sqrt{2}\rangle.$$

The range for v is $0 \le v \le 2\pi$ as usual. For u, we must figure out the height of the cone. The question stipulates that the cone is cut off at $z = 1$. Which u value does this correspond to? Since $z = u\sqrt{2}$, it corresponds to $u = 1/\sqrt{2}$. So, $0 \le u \le 1/\sqrt{2}$.

The tangent vectors are

$$\begin{aligned} \boldsymbol{T_u} &= \langle \cos(v), \sin(v), \sqrt{2}\rangle \\ \boldsymbol{T_v} &= \langle -u\sin(v), u\cos(v), 0\rangle. \end{aligned}$$

Therefore

$$\boldsymbol{N} = \langle -\sqrt{2}u\cos(v), -\sqrt{2}u\sin(v), u\rangle.$$

Stop and check: Is this normal vector pointing upward? Well the z-coordinate is u, which is always non-negative, so yes. If we had done the cross product in the other order it would have been $-u$, which points down, at which point we would negate the whole normal vector.

Now we can integrate:

$$\begin{aligned} \iint_S \langle 0,0,z+y\rangle d\boldsymbol{S} &= \int_0^{1/\sqrt{2}} \int_0^{2\pi} \langle 0,0,z+y\rangle \cdot \langle -\sqrt{2}u\cos(v), -\sqrt{2}u\sin(v), u\rangle \\ &= \int_0^{1/\sqrt{2}} \int_0^{2\pi} \langle 0,0,u\sqrt{2}+u\sin(v)\rangle \cdot \langle -\sqrt{2}u\cos(v), -\sqrt{2}u\sin(v), u\rangle \\ &= \int_0^{1/\sqrt{2}} \int_0^{2\pi} (u^2\sqrt{2} + u^2\sin(v))dvdu \\ &= \dots \\ &= \pi/3. \end{aligned}$$

3. Evaluate $\iint_S yz\, dS$ where S is the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,1,1)$.

   (*Hints:* First you'll want to find an equation for the plane of the form $ax + by + cz = K$. One way to do this is to find a normal vector for the plane: if $\langle A, B, C\rangle$ is orthogonal to the plane, then $Ax + By + Cz = K$ is an equation for the plane (you just have to find the right K). Recall from Math 8 that you can find a normal vector for a plane by finding two vectors that lie in the plane and aren't parallel to each other, then computing their cross product.

   Then, what's the projection of the triangle onto the xy-plane? Just ignore the z-coordinates.)

   For this question, feel free to use a calculator (or Wolfram Alpha) for any tedious expanding or simplifying polynomials.

   Solution: First things first: we need to find a parametrization of the plane. For this we want to write the plane in $ax + by + cz = K$ form. You learn how to do this in Math 8.

   - A plane of the form $ax + by + cz = K$ has normal vector $\langle a, b, c\rangle$.
   - So, if we find the normal vector then we know a, b, and c, and we can use these to find K.

- All we need is *any* vector normal to the plane (with non-zero length). Here's the trick: find two non-parallel vectors in the plane, and take their cross product to get a vector orthogonal to the plane.

One vector in the plane is the vector that goes from $(0,2,0)$ to $(0,1,1)$. This is the vector $\langle 0,-1,1\rangle$. Another is the vector that goes from $(0,2,0)$ to $(1,0,0)$: $\langle 1,-2,0\rangle$. Now,

$$\langle 0,-1,1\rangle \times \langle 1,-2,0\rangle = \langle 2,1,1\rangle.$$

Important: You can't just jump to using this as the normal vector $\boldsymbol{N}$ because we don't have a parametrization yet, and even though this will be parallel to our eventual normal vector, we can't be sure yet if it has the right length.

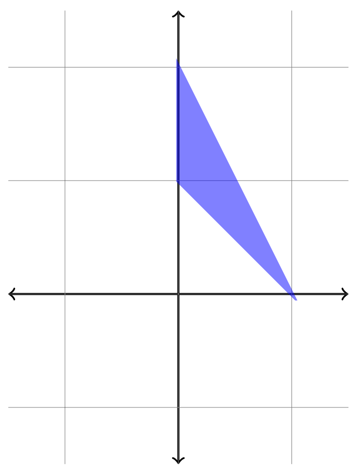
Now we know the plane has the form $2x+y+z=K$, and by plugging in any of the corner points, we find $K=2$. If $2x+y+z=2$, then

$$z = 2-2x-y.$$

Therefore, the plane can be parametrized by

$$G(x,y) = \langle x, y, 2-2x-y\rangle.$$

To figure out the bounds, think about the shadow onto the xy-plane. To project on the xy-plane, all you have to do is ignore the z-coordinates. The three corner points become $(1,0)$, $(0,2)$, and $(0,1)$, which tells us that the projection in the xy-plane is the region shown below.



This is described by the bounds x in $[0,1]$ and y in $[1-x,\ 2-2x]$.

Next we need to find our real normal vector $\boldsymbol{N}$ in the usual way.

$$\boldsymbol{T_x} = \langle 1,0,-2\rangle$$
$$\boldsymbol{T_y} = \langle 0,1,-1\rangle.$$

Therefore,

$$\boldsymbol{N} = \boldsymbol{T_x} \times \boldsymbol{T_y} = \langle 2,1,1\rangle$$

and

$$\|\boldsymbol{N}\| = \sqrt{6}.$$

Note: It won't always work like this, where the normal vector is the same one we already had for the plane.

We have all the parts now, so let's put them together and integrate.

$$
\begin{aligned}
\iint_S yz\,dS &= \sqrt{6}\int_0^1 \int_{1-x}^{2-2x} y(2-2x-y)\,dydx \\
&= \sqrt{6}\int_0^1 \int_{1-x}^{2-2x} (2y-2xy-y^2)\,dydx \\
&= \sqrt{6}\int_0^1 \left[y^2 - xy^2 - \frac{y^3}{3}\right]_{1-x}^{2-2x} dydx \\
&= \sqrt{6}\int_0^1 \left(-\frac{2}{3}x^3 + 2x^2 - 2x + \frac{2}{3}\right) dx \\
&= \frac{1}{\sqrt{6}}.
\end{aligned}
$$