MATH 13, FALL '16

HOMEWORK 7

Due **Tuesday**, Nov 1, **before X-hour**

NO LATE ASSIGNMENTS ACCEPTED

Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. Use a change of coordinates to calculate $\iint_R (x+y)e^{x^2-y^2}\,dA$ where the region R is the parallelogram defined by $0 \le x-y \le 2$ and $0 \le x+y \le 3$. (*Hint:* Recall that $x^2-y^2 = (x+y)(x-y)$.)

   Solution: The question suggests that a good change of coordinates is

   $$u = x - y$$
   $$v = x + y$$

   because then the region of integration is the nice rectangle

   $$0 \le u \le 2,$$
   $$0 \le v \le 3.$$

   By solving for x and y, we find that

   $$x = \frac{u+v}{2}, \qquad y = \frac{v-u}{2}.$$

   The Jacobian of this transformation is

   $$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Hence,

$$\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}\,dA &= \int_0^3 \int_0^2 ve^{uv} \cdot \frac{1}{2}\,dudv \\
&= \frac{1}{2}\int_0^3 \left[e^{uv}\right]_0^2 dv \\
&= \frac{1}{2}\int_0^3 \left(e^{2v}-1\right) dv \\
&= \frac{1}{2}\left[\frac{e^{2v}}{2} - v\right]_0^3 \\
&= \frac{1}{2}\left(\frac{e^6}{2} - 3 - \frac{1}{2}\right) \\
&= \frac{e^6-7}{4}.
\end{aligned}$$

2. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F} = \langle \tan(y), x\sec^2(y)\rangle$ and C is any path from $(1,0)$ to $(2,\pi/4)$. (*Hint:* The derivative of $\tan(y)$ is $\sec^2(y)$.)

   Solution: Since no specific path is given, you should think that $\boldsymbol{F}$ is probably conservative. (Otherwise, you can't do the problem!) You will find that $f = x\tan(y)$ is a potential function. (To check this, just compute ∇f.)

   By the Fundamental Theorem of Conservative Vector Fields,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(2,\pi/4) - f(1,0) = 2\tan(\pi/4) - 1\tan(0) = 2.$$

3. Compute $\int_C xy^2\,ds$ where C is the right half of the circle $x^2+y^2=16$.

   Solution: The curve C can be parametrized by

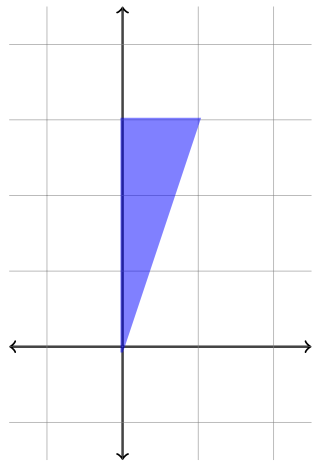
$$\boldsymbol{r}(t) = \langle 4\cos(t),\, 4\sin(t)\rangle, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

   Note that $\boldsymbol{r}'(t) = \langle -4\sin(t),\, 4\cos(t)\rangle$. Now we can compute

$$\begin{aligned}
\int_C x^2y\,ds &= \int_{-\pi/2}^{\pi/2} f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt \\
&= \int_{-\pi/2}^{\pi/2} f(4\cos(t),4\sin(t))\|\langle -4\sin(t),\, 4\cos(t)\rangle\|dt \\
&= \int_{-\pi/2}^{\pi/2} 64\cos(t)\sin^2(t) \cdot \sqrt{16(\sin^2(t)+\cos^2(t))}dt \\
&= 256\int_{-\pi/2}^{\pi/2} \cos(t)\sin^2(t)dt \\
&= 256\left[\frac{\sin^3(t)}{3}\right]_{-\pi/2}^{\pi/2} \\
&= 256\left(\frac{1}{3}+\frac{1}{3}\right) \\
&= \frac{512}{3}.
\end{aligned}$$

4. Use Green's Theorem to compute the **flux** of the vector field $\boldsymbol{F} = \langle x^2y^2, 4xy^3 \rangle$ out of the triangular region with vertices $(0,0)$, $(1,3)$, and $(0,3)$.

Here's a graph of the region:



To compute flux without Green's Theorem, we'd need to compute

$$\int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds$$

over the three curves that make up the boundary. In class we saw that to compute the flux (without Green's Theorem), one can use the formula

$$\int_C \langle -F_2, F_1 \rangle\, d\boldsymbol{r},$$

where $\boldsymbol{F} = \langle F_1, F_2 \rangle$ and C is the boundary oriented counterclockwise. The Divergence form of Green's Theorem says

$$\int_C \langle -F_2, F_1 \rangle\, d\boldsymbol{r} = \iint_R \operatorname{div}(\boldsymbol{F})\, dA.$$

This is easier because it's one double integral of a scalar function instead of three line integrals of vector fields. So,

$$\begin{aligned}
\iint_R \operatorname{div}(\boldsymbol{F})\, dA &= \int_0^1 \int_{3x}^3 \left(2xy^2 + 12xy^2\right) dydx \\
&= \int_0^1 \int_{3x}^3 14xy^2\, dydx \\
&= \frac{14}{3} \int_0^1 \left[xy^3\right]_{3x}^3 dx \\
&= \frac{14}{3} \int_0^1 \left(27x - 27x^4\right) dx \\
&= \frac{14}{3} \left[\frac{27}{2}x^2 - \frac{27}{5}x^5\right]_0^1 \\
&= \frac{14}{3} \cdot \frac{27}{2} - \frac{14}{3} \cdot \frac{27}{5} \\
&= \frac{189}{5}.
\end{aligned}$$