MATH 13, FALL '16

HOMEWORK 6

Due Wednesday, Oct 26, **by 3:00pm**

Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. Let C be the curve parametrized by $\boldsymbol{r}(t) = \langle t, \sqrt{t}, t^2 \rangle$ for $1 \le t \le 4$. Let $\boldsymbol{F} = \langle y+z, x+z, x+y \rangle$.

   a) Without using any results about conservative vector fields, calculate $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

   Solution: From Section 16.2, we know that we calculate line integrals using the formula

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt$$

   Therefore,

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_1^4 \langle y(t)+z(t), x(t)+z(t), x(t)+y(t) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt \\
&= \int_1^4 \left\langle \sqrt{t}+t^2,\, t+t^2,\, t+\sqrt{t} \right\rangle \cdot \left\langle 1,\, \frac{1}{2\sqrt{t}},\, 2t \right\rangle dt \\
&= \int_1^4 \left(\sqrt{t} + t^2 + \frac{\sqrt{t}}{2} + \frac{t^{3/2}}{2} + 2t^2 + 2t^{3/2} \right) dt \\
&= \int_1^4 \left(\frac{3}{2}\sqrt{t} + \frac{5}{2}t^{3/2} + 3t^2 \right) dt \\
&= \left[t^{3/2} + t^{5/2} + t^3 \right]_1^4 \\
&= (8+32+64) - (1+1+1) \\
&= 101.
\end{aligned}$$

   b) Now, calculate this integral again using results about conservative vector fields.

   Solution: A potential function for $\boldsymbol{F}$ is $f(x,y,z) = xy + xz + yz$. You can either find this using the procedure we did in class, or figure out another way and just *verify* that it works. The verification is simple: $\nabla f = \boldsymbol{F}$, so it's a potential function.

   By the *Fundamental Theorem of Conservative Vector Fields*,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(\boldsymbol{r}(4)) - f(\boldsymbol{r}(1)) = f(4,2,16) - f(1,1,1) = 101.$$

2. Evaluate the line integral

$$\int_C \left\langle 6y + \sin\left(x^2\right), 2x^2y + e^{y^2} \right\rangle \cdot d\boldsymbol{r},$$

where C is the circle with radius 1 centered at the origin, oriented **clockwise**.

Solution: This line integral would be difficult or impossible to calculate directly. Green's Theorem requires a counterclockwise curve, so we apply Green's Theorem to $-C$. Using Green's Theorem,

$$\begin{aligned}
\int_C \left\langle 6y + \sin\left(x^2\right), 2x^2y + e^{y^2} \right\rangle \cdot d\boldsymbol{r} &= -\int_{-C} \left\langle 6y + \sin\left(x^2\right), 2x^2y + e^{y^2} \right\rangle \cdot d\boldsymbol{r} \\
&= -\iint_R (4xy - 6)\, dA \\
&= \iint_R (6 - 4xy)\, dA,
\end{aligned}$$

where R is the disc (the inside of a circle) with radius 1 centered at the origin. We've reduced this to a problem that we learned how to do in Chapter 15. By converting the double integral to polar coordinates:

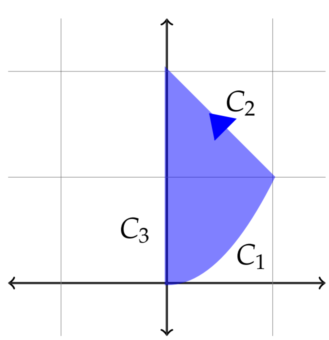
$$\begin{aligned}
\iint_R (6 - 4xy)\, dA &= \int_0^{2\pi} \int_0^1 (6 - 4r^2 \sin(\theta)\cos(\theta)) r\, dr d\theta \\
&= \int_0^{2\pi} \Big[3r^2 - r^4 \sin(\theta)\cos(\theta)\Big]_0^1\, d\theta \\
&= \int_0^{2\pi} (3 - \sin(\theta)\cos(\theta))\, d\theta \\
&= \left[3\theta - \frac{\sin^2(\theta)}{2}\right]_0^{2\pi} \\
&= 6\pi.
\end{aligned}$$

3. Evaluate the line integral of the vector field

$$\boldsymbol{F} = \langle 2y + \cos(x^2), x^2 + y^2 \rangle$$

along the curve that follows the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and then travels along a straight line from $(1,1)$ to $(0,2)$. (*Hint:* Add a line to enclose a region, and think about how Green's Theorem can be applied.)

Let C_1 be the parabola $y = x^2$ from $(0,0)$ to $(1,1)$, let C_2 be the straight line from $(1,1)$ to $(0,2)$, and let C_3 be the straight line from $(0,2)$ to $(0,0)$.



Together, C_1, C_2, and C_3 enclose the region R with a counterclockwise orientation. We want

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

The trick is that we can use Green's Theorem to calculate

$$S = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r}$$

with just a single double integral. Then, if we calculate

$$T = \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r}$$

on its own, we find the answer we want:

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = S - T.$$

Now let's perform these calculations.

By Green's Theorem

$$\begin{aligned}
S &= \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \\
&= \iint_R (2x - 2)\, dA \\
&= \int_0^1 \int_{x^2}^{2-x} (2x-2)\, dy dx \\
&= \int_0^1 \left[(2x-2)y\right]_{x^2}^{2-x}\, dx \\
&= \int_0^1 \big((2x-2)(2-x) - (2x-2)x^2\big)\, dx \\
&= \int_0^1 \big(6x - 4 - 2x^2 - 2x^3 + 2x^2\big)\, dx \\
&= \int_0^1 \big(-2x^3 + 6x - 4\big)\, dx \\
&= \left[-\frac{x^4}{2} + 3x^2 - 4x\right]_0^1 \\
&= -\frac{1}{2} + 3 - 4 \\
&= -\frac{3}{2}.
\end{aligned}$$

Now we compute T by doing the line integral (the old-fashioned way). C_3 is parametrized by $\boldsymbol{r}(t) = \langle 0, 2-t \rangle$ for $0 \leq t \leq 2$.

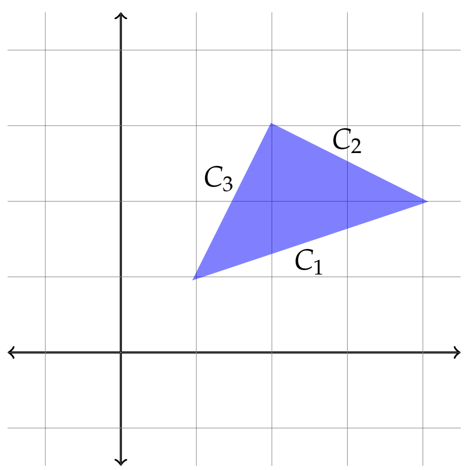
$$
\begin{aligned}
T &= \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \\
&= \int_0^2 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_0^2 \left\langle 2y(t) + \cos(x(t)^2),\, x(t)^2 + y(t)^2 \right\rangle \cdot \langle x'(t), y'(t) \rangle \, dt \\
&= \int_0^2 \left\langle 2(2-t) + \cos(0^2),\, 0^2 + (2-t)^2 \right\rangle \cdot \langle 0, -1 \rangle \, dt \\
&= \int_0^2 \left\langle 5 - 2t,\, 4 - 4t + t^2 \right\rangle \cdot \langle 0, -1 \rangle \, dt \\
&= \int_0^2 \left(-4 + 4t - t^2 \right) \, dt \\
&= \left[-4t + 2t^2 - \frac{t^3}{3} \right]_0^2 \\
&= -\frac{8}{3}.
\end{aligned}
$$

Therefore,

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = S - T = -\frac{3}{2} + \frac{8}{3} = \frac{7}{6}.$$

4. Let D be the triangular region with corners $(1,1)$, $(4,2)$, and $(2,3)$. Use Green's Theorem to calculate the area of D.

   Here's a graph of the region:



   Let R be this region. Define C_1, C_2, and C_3 to be the line segments as pictured above. The area *could* be calculated by splitting the region into two regions and integrating the function 1. (Maybe this would be good practice!) But we're supposed to use Green's Theorem, so we'll do this with three line integrals instead.

   Since we want to find the double integral of 1, we need a vector field $\boldsymbol{F}$ such that $\partial F_2/\partial x - \partial F_1/\partial y = 1$. One such vector field is $\boldsymbol{F} = \langle 0, x \rangle$.

By Green's Theorem:

$$\begin{aligned}
\text{Area} &= \iint_R 1\,dA \\
&= \int_{C_1} \langle 0,\, x\rangle \cdot d\boldsymbol{r} + \int_{C_2} \langle 0,\, x\rangle \cdot d\boldsymbol{r} + \int_{C_{3.}} \langle 0,\, x\rangle \cdot d\boldsymbol{r}
\end{aligned}$$

The parametrizations for the three lines are:

$$\begin{aligned}
C_1 &: \boldsymbol{r}_1(t) = \langle 1+3t,\, 1+t\rangle \\
C_2 &: \boldsymbol{r}_2(t) = \langle 4-2t,\, 2+t\rangle \\
C_3 &: \boldsymbol{r}_3(t) = \langle 2-t,\, 3-2t\rangle
\end{aligned}$$

with bounds $0 \le t \le 1$ for all three.

Hence,

$$\begin{aligned}
\text{Area} &= \int_{C_1} \langle 0,\, x\rangle \cdot d\boldsymbol{r} + \int_{C_2} \langle 0,\, x\rangle \cdot d\boldsymbol{r} + \int_{C_{3.}} \langle 0,\, x\rangle \cdot d\boldsymbol{r} \\
&= \int_0^1 \langle 0,\, 1+3t\rangle \cdot \langle 3,\, 1\rangle\,dt + \int_0^1 \langle 0,\, 4-2t\rangle \cdot \langle -2,\, 1\rangle\,dt + \int_0^1 \langle 0,\, 2-t\rangle \cdot \langle -1,\, -2\rangle\,dt \\
&= \int_0^1 (1+3t)\,dt + \int_0^1 (4-2t)\,dt + \int_0^1 (2t-4)\,dt \\
&= \left[t + \frac{3}{2}t^2\right]_0^1 + \left[4t - t^2\right]_0^1 + \left[t^2 - 4t\right]_0^1 \\
&= \frac{5}{2} + 3 + (-3) \\
&= \frac{5}{2}.
\end{aligned}$$