MATH 13, FALL '16

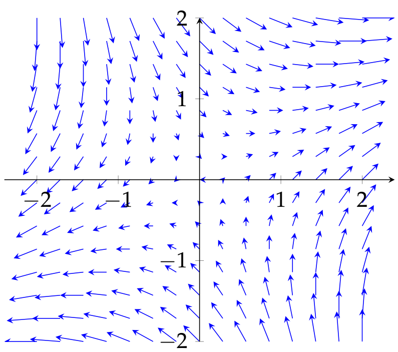
HOMEWORK 5

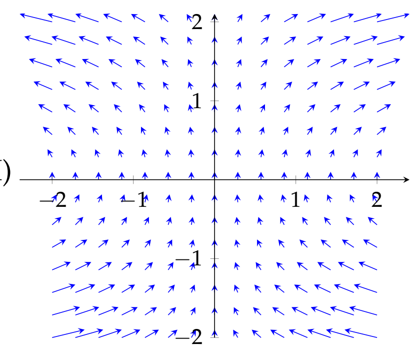
Due Wednesday, Oct 19

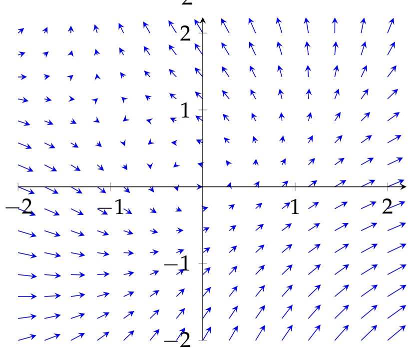
Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

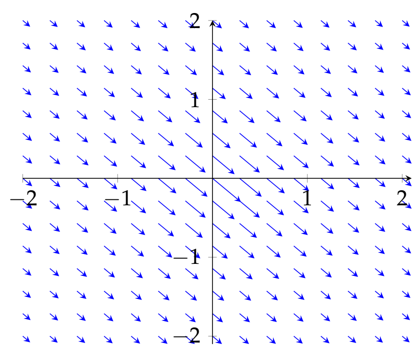
Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. a) Match the vector field with with the plots. **You must justify your answers.**

(I)


(II)


(III)


(IV)


(1) $\langle x^2 - y,\ x + y^2 \rangle$ (2) $\langle xy,\ 1 \rangle$

(3) $\langle x + y,\ x - y \rangle$ (4) $\left\langle \dfrac{1}{\sqrt{x^2+y^2+1}},\ -\dfrac{1}{\sqrt{x^2+y^2+1}} \right\rangle$

Solution: Use process of elimination. First, observe that whichever vector field is $\langle xy, 1 \rangle$ should have *every* arrow with an upward magnitude of 1 regardless of the left-to-right direction. Only (II) has this property.

Another good technique is to pick particular points. For example, at $(x, y) = (0, 0)$ vector fields (1) and (3) should have value $\langle 0, 0 \rangle$ while vector field (4) should have value $\langle 1, -1 \rangle$. This tells is that (4) matches (IV), and we're left only with matching (1) and (3) with (I) and (III).

At $(x, y) = (1, 1)$, the vector field (1) has value $\langle 0, 2 \rangle$ while the vector field (2) has value $\langle 2, 0 \rangle$. Therefore, we see that (1) matches (III) and (3) matches (I).

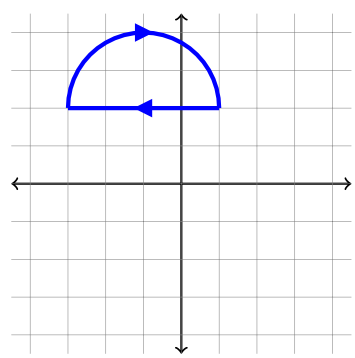
b) Find the curl and divergence of the vector field $\langle x^2 - y, x + y^2, 2x \rangle$.

Solution:

$$\begin{aligned} \operatorname{curl}(\boldsymbol{F}) &= \boldsymbol{\nabla} \times \boldsymbol{F} \\ &= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \\ &= \langle 0, -2, 2 \rangle. \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\boldsymbol{F}) &= \boldsymbol{\nabla} \cdot \boldsymbol{F} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= 2x + 2y. \end{aligned}$$

2. Let $\boldsymbol{F} = \langle 2y, -3x \rangle$, and let $\mathcal{C}$ be the curve below. The curved component is half of a circle. Find $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$. *Hint:* $\int \sin^2(t)dt = \frac{t}{2} - \frac{\sin(2t)}{4} + C$ and $\int \cos^2(t)dt = \frac{t}{2} + \frac{\sin(2t)}{4} + C$.



Solution: The first step is to parametrize the curve. Let $\mathcal{C}_1$ be the lower straight edge and let $\mathcal{C}_2$ be the upper half-circle. Since $\mathcal{C}_1$ goes from $(1, 2)$ to $(-3, 2)$, we can it parametrize by

$$\mathcal{C}_1 : \boldsymbol{r_1}(t) = \langle 1 - t,\, 2 \rangle, \quad 0 \le t \le 4.$$

The upper half circle has radius 2 and is centered at $(-1, 2)$. If we were parametrizing the full circle in the usual counterclockwise orientation, that would be

$$\langle 2\cos(t) - 1,\, 2\sin(t) + 2 \rangle, \quad 0 \le t \le 2\pi.$$

To adapt this for our upper half-circle, oriented clockwise, we have two options. The first is to parametrize with the counterclockwise orientation instead, and then remember to make its corresponding integral negative by swapping the bounds. The second is to figure out how to swap sin and cos and/or add negative signs to get the correct half-circle. To do it this second way, observe that if we flipped the upper half-circle across a horizontal line, we'd get the *bottom half-circle*, with the normal orientation. So,

$$\mathcal{C}_2 : \boldsymbol{r_2}(t) = \langle 2\cos(t) - 1,\, -2\sin(t) + 2 \rangle, \quad \pi \le t \le 2\pi.$$

(When in doubt, just try some things, then plot some points and see if it's right! In this case, just plug in $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$ to verify correctness.)

Now we calculate the integrals, using the fact that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

For the first integral, we see

$$\begin{aligned}\int_{C_1} \boldsymbol{F} \cdot \boldsymbol{r} &= \int_0^4 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\ &= \int_0^4 \langle 2y(t),\, -3x(t)\rangle \cdot \langle -1,\, 0\rangle\, dt \\ &= \int_0^4 \langle 4, 3t-3\rangle \cdot \langle -1,\, 0\rangle\, dt \\ &= \int_0^4 -4dt \\ &= -16.\end{aligned}$$

For the second integral, we see

$$\begin{aligned}\int_{C_2} \boldsymbol{F} \cdot \boldsymbol{r} &= \int_\pi^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\ &= \int_\pi^{2\pi} \langle 2y(t),\, -3x(t)\rangle \cdot \langle -2\sin(t),\, -2\cos(t)\rangle\, dt \\ &= \int_\pi^{2\pi} \langle 4 - 4\sin(t),\, 3 - 6\cos(t)\rangle \cdot \langle -2\sin(t),\, -2\cos(t)\rangle\, dt \\ &= \int_\pi^{2\pi} (8\sin^2(t) - 8\sin(t) + 12\cos^2(t) - 6\cos(t))dt \\ &= [(4t - 2\sin(2t)) + 8\cos(t) + (6t + 3\sin(2t)) - 6\sin(t)]_\pi^{2\pi} \\ &= [10t + \sin(2t) + 8\cos(t) - 6\sin(t)]_\pi^{2\pi} \\ &= (20\pi + \sin(4\pi) + 8\cos(2\pi) - 6\sin(2\pi)) - (10\pi + \sin(2\pi) + 8\cos(\pi) - 6\sin(\pi)) \\ &= (20\pi + 0 + 8 - 0) - (10\pi + 0 - 8 - 0) \\ &= 10\pi + 16.\end{aligned}$$

Therefore,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = 10\pi.$$

3. a) Find a potential function for $\boldsymbol{F}$ or prove that $\boldsymbol{F}$ is not conservative.

$$\boldsymbol{F} = \left\langle y\ln(z) + ye^{xy},\ x\ln(z) + xe^{xy},\ \frac{xy}{z} + 2z \right\rangle$$

Solution: Since it looks like computing the curl of $\boldsymbol{F}$ would be painful, let's first try to find a potential function instead. Then, we only have to compute the curl if we fail to find a potential function.

Suppose that there is a potential function $f(x,y,z)$. This implies that

$$\frac{\partial f}{\partial x} = y\ln(z) + ye^{xy},$$
$$\frac{\partial f}{\partial y} = x\ln(z) + xe^{xy},$$
$$\frac{\partial f}{\partial z} = \frac{xy}{z} + 2z.$$

Let's see what information we can derive about $f(x,y,z)$ with this information.

If $\frac{\partial f}{\partial x} = y\ln(z) + ye^{xy}$ then $f(x,y,z) = xy\ln(z) + e^{xy} + g(y,z)$, where $g(y,z)$ is a function with no x. (This is like the usual $+C$ when you integrate, but here it's a constant with respect to x, which means it's allowed to have the other variables in it.)

Now if $f(x,y,z) = xy\ln(z) + e^{xy} + g(y,z)$, then $\frac{\partial f}{\partial y} = x\ln(z) + xe^{xy} + \frac{\partial g}{\partial y}$. Since we already know that $\frac{\partial f}{\partial y} = x\ln(z) + xe^{xy}$, it follows that $\frac{\partial g}{\partial y} = 0$. Therefore, $g(y,z) = 0 + h(z)$, where $h(z)$ is a function with no x and y in it.

Now if $f(x,y,z) = xy\ln(z) + e^{xy} + h(z)$, then $\frac{\partial f}{\partial z} = \frac{xy}{z} + \frac{\partial h}{\partial z}$. Since we already know that $\frac{\partial f}{\partial z} = \frac{xy}{z} + 2z$, we conclude that $\frac{\partial h}{\partial z} = 2z$. Therefore, $h(z) = z^2$.

Combining this, we get that

$$f(x,y,z) = xy\ln(z) + e^{xy} + z^2$$

is a potential function for $\boldsymbol{F}$. (We're also allowed to add any scalar constant c and we still have a potential function.)

If you chose to actually compute the curl to try to argue that since the curl is zero, it must be conservative, then you *must point out* that the domain of $\boldsymbol{F}$ is $\{(x,y,z) : z > 0\}$ and that this domain is simply connected. Without this hypothesis, it's not true that zero curl implies conservative. In fact, we won't even learn that you can do this until 16.3.

b) Find a potential function for $\boldsymbol{F}$ or prove that $\boldsymbol{F}$ is not conservative.

$$\boldsymbol{F} = \left\langle x^2\sin(y),\ \frac{x^3\cos(y)}{3} + 2yz,\ y \right\rangle$$

This one actually doesn't look too bad to take the curl of, so let's try that. The x-component of $\operatorname{curl}(\boldsymbol{F})$ is

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 1 - 2y,$$

and now we can stop. Since $1 - 2y$ is not identically equal to zero everywhere in the domain, the vector field cannot be conservative.

4. Compute $\int_{\mathcal{C}} (2x + 9z)\, ds$ where $\mathcal{C}$ is the curve $\boldsymbol{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \leq t \leq 1$.

Solution: The curve is already parametrized for us, so we just use our formula for computing line integrals.

$$\begin{aligned} \int_{\mathcal{C}} (2x + 9z)\, ds &= \int_0^1 (2x(t) + 9z(t)) \|\boldsymbol{r}'(t)\| dt \\ &= \int_0^1 (2t + 9t^3) \|\langle 1, 2t, 3t^2 \rangle\| dt \\ &= \int_0^1 (2t + 9t^3) \sqrt{1 + 4t^2 + 9t^4}\, dt \end{aligned}$$

This is perfectly suited for u-substitution. Let $u = 1 + 4t^2 + 9t^4$. Then

$$du = (8t + 36t^3)dt = 4(2t + 9t^3)dt.$$

Therefore,

$$\begin{aligned} \int_0^1 (2t + 9t^3) \sqrt{1 + 4t^2 + 9t^4}\, dt &= \left[\frac{1}{4} \cdot \frac{2}{3} \left(1 + 4t^2 + 9t^4 \right)^{3/2} \right]_0^1 \\ &= \frac{1}{6} \left(14^{3/2} - 1 \right). \end{aligned}$$