MATH 13, FALL '16

HOMEWORK 4

Due Wednesday, October 12

Write your answers neatly and clearly. Use complete sentences, and label any diagrams. List problems in numerical order and staple all pages together. Start each problem on a new page. Please show your work; no credit is given for solutions without work or justification.

Remember that you may discuss the problems with classmates, but all work should be your own. List the names of anybody with whom you discussed the problems at the top of the page.

1. Find the mass and center of mass of the solid cube $\mathcal{W} = [0, C] \times [0, C] \times [0, C]$ with density $\rho(x, y, z) = x^2 + y^2 + z^2$.

   Solution: First we note that since the density function ρ is symmetric (in the sense that if we interchange any variables we get the same function), and the region has the same bounds for each variable, then the x, y, and z coordinates of the center of mass will all be the same.

   The x-coordinate of the center of mass is

$$x_{\mathrm{CM}} = \frac{M_y}{M}$$

   where M is the total mass and

$$M_y = \iiint_W x\rho(x, y, z)\, dV.$$

   The total mass is:

$$\begin{aligned}
M &= \int_0^C \int_0^C \int_0^C (x^2 + y^2 + z^2)\, dzdydx \\
&= \int_0^C \int_0^C \left[z(x^2 + y^2) + \frac{z^3}{3} \right]_0^C dydx \\
&= \int_0^C \int_0^C \left(C(x^2 + y^2) + \frac{C^3}{3} \right) dydx \\
&= \int_0^C \left[Cx^2y + \frac{Cy^3}{3} + \frac{C^3}{3}y \right]_0^C dx \\
&= \int_0^C \left(C^2x^2 + \frac{2C^4}{3} \right) dx \\
&= \left[\frac{C^2x^3}{3} + \frac{2C^4x}{3} \right]_0^C \\
&= C^5
\end{aligned}$$

Next we calculate M_y:

$$\begin{aligned}
M_y &= \int_0^C \int_0^C \int_0^C x(x^2+y^2+z^2)\,dzdydx \\
&= \int_0^C \int_0^C \left[z(x^3+xy^2) + \frac{xz^3}{3} \right]_0^C dydx \\
&= \int_0^C \int_0^C \left(C(x^3+xy^2) + \frac{xC^3}{3} \right) dydx \\
&= \int_0^C \left[Cx^3y + \frac{Cxy^3}{3} + \frac{xC^3}{3}y \right]_0^C dx \\
&= \int_0^C \left(C^2x^3 + \frac{2xC^4}{3} \right) dx \\
&= \left[\frac{C^2x^4}{4} + \frac{C^4x^2}{3} \right]_0^C \\
&= \frac{7C^6}{12}.
\end{aligned}$$

Therefore,

$$x_{\text{CM}} = \frac{7C^6/12}{C^5} = \frac{7C}{12}.$$

By the symmetry argument above, the center of mass is

$$(x,y,z) = \left(\frac{7C}{12}, \frac{7C}{12}, \frac{7C}{12} \right).$$

2. Let $\mathcal{D}$ be a two-dimensional region that occupies the region that lies outside the circle $x^2+y^2=1$ and inside the circle $x^2+y^2=2y$. Suppose this region has density function $\delta(x,y) = \frac{k}{\sqrt{x^2+y^2}}$. (In other words, the density is inversely proportional to the distance from the origin.) What is the center of mass?

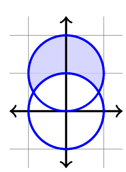
Solution: The circle

$$x^2+y^2=2y$$

which is

$$x^2+(y-1)^2=1.$$

Now we draw the region.



The lower circle is $r=1$ and the upper circle is $r=2\sin(\theta)$. They intersect at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Converted to polar, the density function is $\delta(r,\theta) = k/r$.

The total mass is

$$
\begin{aligned}
M &= \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin(\theta)} k \, dr d\theta \\
&= k \int_{\pi/6}^{5\pi/6} (2\sin(\theta) - 1) \, d\theta \\
&= k \left[-2\cos(\theta) - \theta\right]_{\pi/6}^{5\pi/6} \\
&= k \left((-2\cos(5\pi/6) - 5\pi/6) - (-2\cos(\pi/6) - \pi/6)\right) \\
&= k((\sqrt{3} - 5\pi/6) + (\sqrt{3} + \pi/6) \\
&= k(2\sqrt{3} - 2\pi/3).
\end{aligned}
$$

We now calculate M_x and M_y:

$$
\begin{aligned}
M_y &= \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin(\theta)} kr\cos(\theta) \, dr d\theta \\
&= k \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\cos(\theta)\right]_1^{2\sin(\theta)} d\theta \\
&= k \int_{\pi/6}^{5\pi/6} \left(2\sin^2(\theta)\cos(\theta) - \frac{1}{2}\cos(\theta)\right) d\theta \\
&= k \left[\frac{2}{3}\sin^3(\theta) - \frac{1}{2}\sin(\theta)\right]_{\pi/6}^{5\pi/6} \\
&= k \left(\left(\frac{2}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{4}\right) - \left(\frac{2}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{4}\right)\right) \\
&= 0.
\end{aligned}
$$

(*Note:* You may have seen that $M_y = 0$ by symmetry, but if not you can always just integrate!)

Next, M_x:

$$
\begin{aligned}
M_x &= \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin(\theta)} kr\sin(\theta) \, dr d\theta \\
&= k \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\sin(\theta)\right]_1^{2\sin(\theta)} d\theta \\
&= k \int_{\pi/6}^{5\pi/6} \left(2\sin^3(\theta) - \frac{1}{2}\sin(\theta)\right) d\theta \\
&= k \int_{\pi/6}^{5\pi/6} \left(2\sin(\theta) - 2\sin(\theta)\cos^2(\theta) - \frac{1}{2}\sin(\theta)\right) d\theta \\
&= k \left[-\frac{3}{2}\cos(\theta) + \frac{2}{3}\cos^3(\theta)\right]_{\pi/6}^{5\pi/6} \\
&= k \left(\left(\left(-\frac{3}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{2}{3}\right)\left(-\frac{\sqrt{3}}{2}\right)^3\right) - \left(\left(-\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)^3\right)\right) \\
&= 2k\left(\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) \\
&= k\sqrt{3}.
\end{aligned}
$$

Hence, the center of mass is

$$\begin{aligned}
\text{COM} &= (x_{\text{CM}}, y_{\text{CM}}) \\
&= \left(\frac{M_y}{M}, \frac{M_x}{M}\right) \\
&= \left(0, \frac{k\sqrt{3}}{k(2\sqrt{3} - 2\pi/3)}\right) \\
&= \left(0, \frac{\sqrt{3}}{2\sqrt{3} - 2\pi/3}\right).
\end{aligned}$$

3. Let $\mathcal{R}$ be the parallelogram with vertices $(-1,3), (1,-3), (3,-1), (1,5)$. Use the transformation $x = \dfrac{u+v}{4}$ and $y = \dfrac{v-3u}{4}$ to calculate $\iint_{\mathcal{R}} (4x+8y)\, dA$.

   Solution: We're given the transformation $(x(u,v) = (u+v)/4$ and $y(u,v) = (v-3u)/4)$, but we need to determine what parallelogram in xy-coordinates turns into the given parallelogram. (If we're lucky, it's a rectangle that becomes this parallelogram!)

   We need to do a few things:

   a) Convert the given parallelogram in xy-coordinates to (hopefully!) a rectangle in uv-coordinates.

   b) Convert the integrand from xy-coordinates to uv-coordinates.

   c) Find the Jacobian (correction factor).

   d) Put it all together and integrate.

   One way to do this is to solve the given transformation for u and v. Since $x = (u+v)/4$, it follows that $v = 4x - u$. Since $y = (v-3u)/4$, it follows that $v = 4y + 3u$. Hence

$$4x - u = 4y + 3u,$$

   and thus

$$u = x - y.$$

   Plugging into the equation $v = 4x - u$,

$$v = 3x + y.$$

   Hence, the corners $(-1,3), (1,-3), (3,-1), (1,5)$ in the xy-plane correspond to the corners $(-4,0), (4,0), (4,8), (-4,8)$ in the uv-plane.

   The integrand $4x + 8y$ becomes $(u+v) + (2v - 6u) = 3v - 5u$.

   The Jacobian is

$$\begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}.$$

So,

$$\begin{aligned}
\iint_{\mathcal{R}} (4x+8y)\,dA &= \int_0^8 \int_{-4}^4 (3v-5u)\left(\frac{1}{4}\right)\,dudv \\
&= \frac{1}{4}\int_0^8 \left[3vu - \frac{5u^2}{2}\right]_{-4}^{4} dv \\
&= \frac{1}{4}\int_0^8 ((12v-40)-(-12v-40))\,dv \\
&= \frac{1}{4}\int_0^8 (24v)\,dv \\
&= \frac{1}{4}\left[12v^2\right]_0^8 \\
&= \frac{12\cdot 64}{4} \\
&= 192.
\end{aligned}$$

4. Let $\mathcal{R}$ be the region bounded by the curves $xy=1$, $xy=2$, $xy^2=1$, and $xy^2=2$. Graph the region (you may use a computer to help). Use the transformation $u=xy$ and $v=xy^2$ to calculate $\iint_{\mathcal{R}} y^2\,dA$.

   Solution: Here the rectangle is given to us:

$$\begin{array}{ccccc} 1 & \le & xy & \le & 2, \\ 1 & \le & xy^2 & \le & 2. \end{array}$$

   To find the integrand we solve for x and y. By the two equations, $x=u/y$ and $x=v/y^2$. Therefore, $u/y=v/y^2$, and so $y=v/u$. Thus, $x=u^2/v$.

   The integrand y^2 is $\dfrac{v^2}{u^2}$.

   The Jacobian is

$$\begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{2}{v} - \frac{1}{v} = \frac{1}{v}.$$

   So,

$$\begin{aligned}
\iint_{\mathcal{R}} y^2\,dA &= \int_1^2 \int_1^2 \frac{v^2}{u^2}\cdot\frac{1}{v}dudv \\
&= \left(\int_1^2 v\,dv\right)\left(\int_1^2 \frac{1}{u^2}\,du\right) \\
&= \left[\frac{v^2}{2}\right]_1^2 \cdot \left[-\frac{1}{u}\right]_1^2 \\
&= \left(2-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right) \\
&= \frac{3}{4}.
\end{aligned}$$