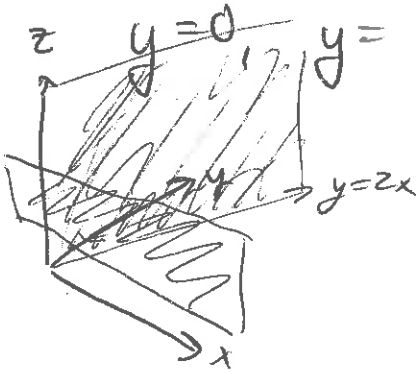
Q1: 5pts
Q2: 5pts
Q3: 5pts
Q4: 5pts

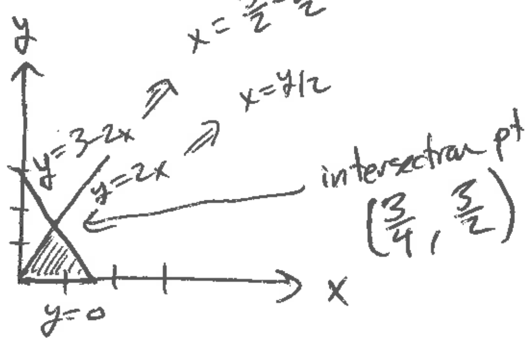
Math 13, Homework 2 Solutions

1 a) Use double integrals to find the volume of the figure enclosed by the planes $z=0$, $y=0$, $y=2x$, $4x+2y+z=6$



Region in the xy-plane bounded by $y=2x$, $y=0$, and the line where $4x+2y+z=6$ intersects the xy-plane (i.e. with $z=0$), which is $4x+2y=6$

$$\Rightarrow y = 3-2x.$$



Region: $\begin{cases} y \text{ in } \left[0, \frac{3}{2}\right] \\ x \text{ in } \left[\frac{y}{2}, \frac{3}{2}-\frac{y}{2}\right] \end{cases}$

$$\text{Volume} = \int_{y=0}^{y=3/2} \int_{x=y/2}^{x=\frac{3}{2}-\frac{y}{2}} (6-4x-2y)\,dx\,dy$$

top surface: $z = 6-4x-2y$
bottom surface: $z=0$

$$= \int_{y=0}^{y=3/2} \left[6x-2x^2-2xy\right]_{x=y/2}^{x=3/2-y/2} dx = \int_{y=0}^{y=3/2} \left[\left((6-2y)\left(\tfrac{3}{2}-\tfrac{y}{2}\right)-2\left(\tfrac{3}{2}-\tfrac{y}{2}\right)^2\right) - \left(3y-\tfrac{y^2}{2}-y^2\right)\right] dy$$

$$= \int_{y=0}^{y=3/2} \left[9 - 3y - 3y + y^2 - 2\left(\frac{9}{4} - \frac{3y}{2} + \frac{y^2}{4}\right) - 3y + \frac{y^2}{2} + y^2 \right] dy$$

$$= \int_{y=0}^{y=3/2} \left(y^2\left(1 - \tfrac{1}{2} + \tfrac{1}{2} + 1\right) + y\left(-6 + 3 - 3\right) + \left(9 - \tfrac{9}{2}\right)\right) dy$$

$$= \int_{y=0}^{y=3/2} \left(2y^2 - 6y + 9/2\right) dy = \left[\frac{2}{3}y^3 - 3y^2 + \frac{9}{2}y\right]_{y=0}^{y=3/2}$$

$$= \frac{2}{3}\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)^2 + \frac{9}{2}\cdot\frac{3}{2}$$

$$= \frac{9}{4} - \frac{27}{4} + \frac{27}{4} = \boxed{\frac{9}{4}}$$

b) What is the average height of a point in this region?

Let $h(x,y) = 6 - 4x - 2y$. Then,

$$\bar{h} = \frac{1}{\text{Area}(D)} \iint_D h \, dA,$$

where D is the triangular region in the xy-plane drawn above.

Since D is a triangle, we can calculate its area without calculus. $\text{Area}(D) = \frac{1}{2} \cdot \text{width} \cdot \text{height}$

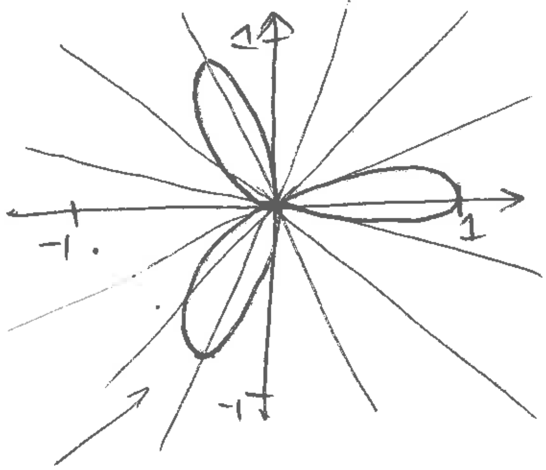
$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2}$$

$$= \frac{9}{8}.$$

So, the average height is

$$\bar{h} = \frac{1}{9/8} \cdot \frac{9}{4} = \frac{8}{9} \cdot \frac{9}{4} = 2.$$

2 a) Plot $r = \cos(3\theta)$ for $0 \le \theta \le 2\pi$.



This is $0 \le \theta \le \pi$,
then it repeats again.

Note $r = 0$ when

$$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

so $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

$r = 1$ when

$$3\theta = 0, 2\pi, 4\pi, \ldots$$

$\Rightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

$r = -1$ when

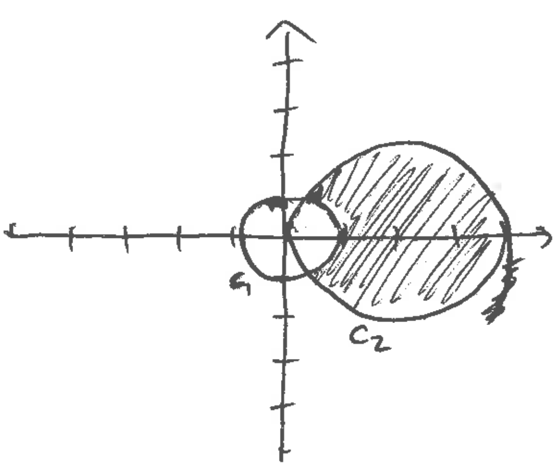
$$3\theta = \pi, 3\pi, 5\pi$$

$$\theta = \pi/3, \pi, 5\pi/3$$

b) Express the region that lies inside C_1 and outside C_2 in polar coordinates.

$C_1 = 1$

$C_2 = 2\cos(\theta)$



intersection points:

$C_1 = C_2$

$1 = 2\cos(\theta)$

(we can do this because the intersection points are not at the origin!)

$\frac{1}{4} = \cos(\theta)$

$\Rightarrow \theta = -\arccos\left(\frac{1}{4}\right), \arccos\left(\frac{1}{4}\right)$

$$\begin{cases} \theta \text{ in } [-\arccos(\tfrac{1}{4}), \arccos(\tfrac{1}{4})] \\ r \text{ in } [1, 4\cos(\theta)] \end{cases}$$

c) Set up, but <u>do not evaluate</u>, the integral of $f(x,y) = x + 2y$ over the region from (b).

Convert to polar form $\rightarrow$

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$x+2y \rightarrow r(\cos(\theta) + 2\sin(\theta))$$

so:

$$\int_{\theta=-\arccos(\frac{1}{4})}^{\theta=\arccos(\frac{1}{4})} \int_{r=1}^{r=4\cos(\theta)} r^2 (\cos\theta + 2\sin(\theta))\, dr\, d\theta$$

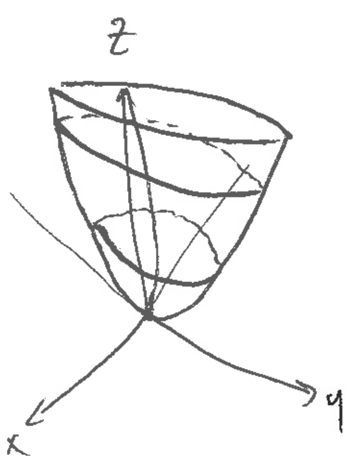
correction factor

3) Use polar integration to find the volume of the figure above the xy-plane, under the paraboloid $z=2x^2+y^2$, and inside the cylinder $x^2+y^2=2y$.

Let's graph some things first.

$x^2+y^2=2y$ has no z in it, so the shape has a curve in the xy-plane ($x^2+y^2=2y$) and all z-points above and below it. (Just like in two variables, the curve $x=3$ has no y, so it's the point 3 and all y-values above and below it.)

$z=2x^2+y^2$



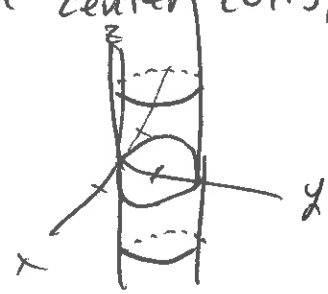
Note that $x^2+y^2=2y$

$$\Rightarrow x^2+y^2-2y=0$$

$$\Rightarrow x^2+y^2-2y+1=1$$

$$\Rightarrow x^2+(y-1)^2=1$$

Circle with center $(0,1)$, radius 1.



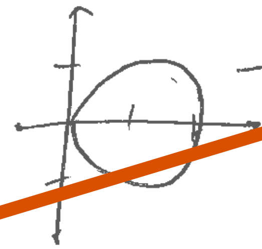
So, we want the volume between $z=0$ and $z=2x^2+y^2$ over the region inside $x^2+(y-1)^2=1$.

the projection onto the xy plane is
r = 2*sin(theta)

Convert to polar:

$2x^2+y^2 \to 2r^2\cos^2(\theta)+r^2\sin^2(\theta)$

$= r^2(\cos^2(\theta)+1)$



~~this is $r=2\cos(\theta)$, $0 \le \theta \le \pi$.~~

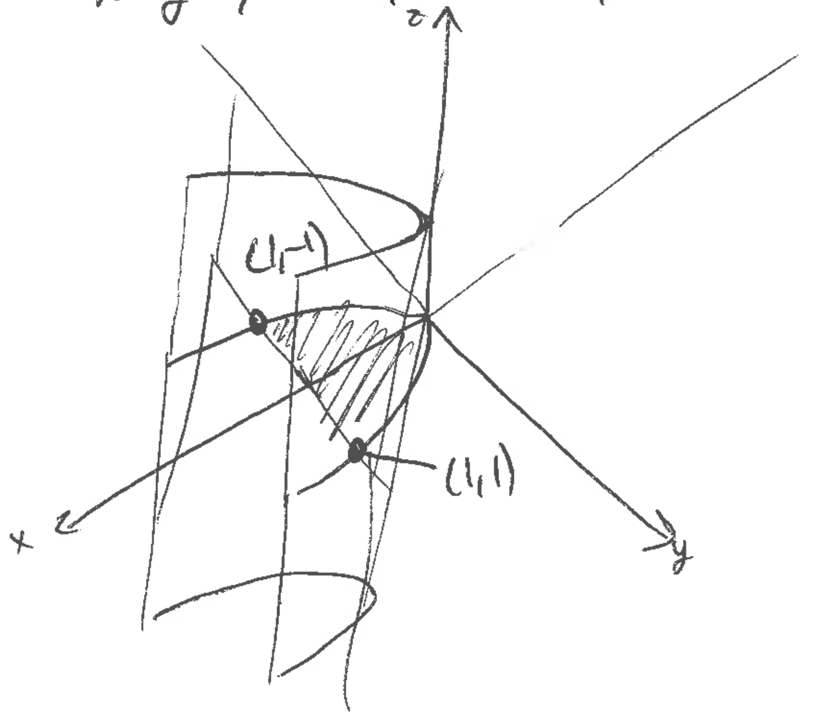
If you use 2π, you'll get double!

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{\overset{r=2\sin(\theta)}{\cancel{r=2\cos(\theta)}}} r^2(1+\cos^2(\theta))\, r\, dr\, d\theta$$

correction factor

Do not evaluate.

4) Find the integral of the function $xy+z$ over the region bounded by the surfaces $x=y^2$, $x=1$, $z=x$, $z=-x$.



$$\int_{y=-1}^{y=1}\int_{x=y^2}^{x=1}\int_{z=-x}^{z=x}(xy+z)\,dz\,dx\,dy$$

$$=\int_{y=-1}^{y=1}\int_{x=y^2}^{x=1}\left[xyz+\frac{z^2}{2}\right]_{z=-x}^{z=x}dx\,dy=\int_{y=-1}^{y=1}\int_{x=y^2}^{x=1}\left(\left(x^2y+\frac{x^2}{2}\right)-\left(-x^2y+\frac{x^2}{2}\right)\right)dx\,dy$$

$$= \int_{y=-1}^{y=1} \int_{x=y^2}^{x=1} 2x^2 y \, dx \, dy = \int_{y=-1}^{y=1} \left[\frac{2}{3}x^3 y\right]_{x=y^2}^{x=1} dy$$

$$= \int_{y=-1}^{y=1} \left(\frac{2}{3}y - \frac{2}{3}y^7\right) dy = \left[\frac{1}{3}y^2 - \frac{1}{12}y^8\right]_{y=-1}^{y=1}$$

$$= \left(\frac{1}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{12}\right)$$

$$= 0.$$